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Association of two square difference identity to regular polygons and circles

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ABSTRACT

GeoGebra is a dynamic software that is frequently used and of increasing importance in mathematics teaching processes in our digital age. Accordingly, in this study a new perspective has been brought to the proofs of the "two square difference identity" expressed for the square, which is a flat polygon, made with different approaches. With side lengths a, b, and a>b, it has been shown that the identity given by the equation (difference of area) $a^2-b^2=(a-b)(a+b)$ is true for other regular polygons as well. In the study, direct proof method was used within the framework of the principle of conservation of measure, which is one of the basic principles of geometry teaching. GeoGebra program, which is a dynamic geometry software, was preferred for drawing geometric shapes used in proofs. In order to generalize the number n, a different fragmentation technique was preferred to the proofs made using different drawings for equilateral triangle and square, which are the simplest regular polygons. It has also been shown that this identity is true for circles viewed as polygons with an infinite number of sides.

Keywords: association, geometry teaching, identities, difference of two squares, visualization, dynamic geometry software, GeoGebra

INTRODUCTION

Visualization & Geometry Education

Visualization is one of the prominent concepts in teaching geometric concepts and expressing algebraic concepts with geometric shapes. Thanks to visualization, geometric thinking comes to the fore. Duval (1998) evaluates geometric thinking in a three-stage cognitive process: Visualization, formation, and reasoning. The first step of this cognitive process, visualization, is the visual representation of geometric expressions or the intuitive or experimental exploration of a complex geometric condition. As a matter of fact, as Kose (2008) states, most of the geometry research are based on the visualization of geometric concepts by drawing their shapes and making some generalizations based on these visuals. Visualization plays highly important role in discovering mathematical concepts and revealing the relationships between these concepts (Hamersma, 2002). For this reason, visualization is frequently used in mathematics education. One reason for this can be thought of as the complex structure of mathematics. Because visualization allows to reduce complexity while dealing with a large amount of information (Rösken & Rolka, 2006).

Thanks to visualization, many skills can be gained in relation to mathematics itself, to other concepts or to daily life. National Council of Teachers of Mathematics (NCTM, 2000), an international organization on this subject, stated that the subject areas of mathematics are not independent from each other, on the contrary, it is an integrated field of study. In addition, similar results and comments can be found in the literature. Yulianto et al. (2019) expressed associating with visualizations as a kind of skill that helps to establish connections between different mathematical concepts. Putri and Santosa (2015) describe the ability to relate, which is closely related to visualization, as the ability to connect mathematical concepts with the learner's daily life or other mathematical concepts. Again, According to Yulianto et al. (2019) visualization in mathematics education plays an important role in the effort to solve problem situations encountered in daily life, and with this feature, mathematics education and visualization lead to the emergence of the ability to relate.

Visualization plays an important role in concretizing the abstract structure of mathematics in the mind and creating mental schemas. Visualizations, which are frequently used in mathematics education, aim to concretize some abstract mathematical concepts that are difficult to understand and to increase the level of interaction with these concepts (Liang & Sedig, 2010). Ozdemir et al. (2005) stated that the use of visualization in the teaching processes, which allows students to see details better in the education system, will contribute to a better understanding of mathematics and increase success in mathematics. English and

Watters (2005) stated that problems solved using visuals are important in developing metacognitive and critical thinking skills. Borromeo-Ferri (2006) states that mathematical results should be reconciled with real situations using visuals and turned into real results. Presmeg (2006) and Stylianou and Silver (2004) stated that visuals, which have an important place in thinking processes, have an important contribution to learning and understanding. Rahim and Siddo (2009) stated that visual thinking and visualization should not only be limited to the field of geometry but should also be used to make sense of mathematical processes and concepts.

Visuals and visualizations are frequently used while proving the propositions put forward in mathematics. While the proof processes were carried out intensively with algebraic expressions in the past, today they are also carried out with visuals together with technological developments. At this point, Hanna and Williers (2008) stated that proofs in mathematics are generally done algebraically, that many students do not fully understand the reason for these proofs, and that visualization supports the proof process at this point. In the 20th century, which is a recent period, the habit of drawing in the proof process has decreased and proofs have been perceived only as a set of algebraic operations. In recent times, as we approach today, "verbal proofs" has become important and has been featured in scientific journals a lot with visuals (Alsina & Nelsen, 2006). Thanks to the technological developments realized at this point, the materials created with dynamic software played an important role in bringing clearer, precise and meaningful proofs to the fore. From these points, in this study, it is considered important to express and prove the identity of the difference of two squares, which is expressed for the square, which is itself a regular polygon, for regular polygons with n sides, especially equilateral triangle and regular pentagon. In addition to the fragmentation used in the proofs for the square, a visual proof is made using a new fragmentation method used for the regular polygon with n sides.

Visualization & Proof with GeoGebra

For learning processes, visualization is extremely important for concretizing abstract data. At this point, in addition to concrete objects in classroom learning, digital objects, platforms and software come to the fore in the 21st century. Technological developments have made it possible to use dynamic visualization in mathematics learning classrooms (Zhang et al., 2023). Many studies have noted the link between the integration of dynamic visualization into instruction and students' progress and performance in learning mathematics (Liang & Sedig, 2010; Ziatdinov & Valles, 2022). GeoGebra is an open source dynamic visualization software package used in many studies. (Zulnaidi et al., 2020). GeoGebra and dynamic visualizations were found to play an important role in providing external representation, deepening information processing, engaging students' interest, problem solving and innovative thinking (Granberg & Olsson, 2015). Chan and Leung (2014) conducted a systematic meta-analysis and concluded that teaching based on dynamic geometry software such as GeoGebra is important for improving students' mathematics achievement. More recent studies have also shown that translating between representations through digital software, such as GeoGebra, can lead to higher levels of student understanding of mathematical concepts (Johnson, 2022; Kohen et al., 2019; Zulnaidi et al., 2020). Kohen et al. (2019) stated that dynamic visualizations can be adopted as tools to concretize learning materials and ensure students' active participation. GeoGebra has also been shown to be applicable to mathematics teaching and learning (Baye et al., 2021). Dynamic geometry software such as GeoGebra contributes to and improves the ability to make proofs as well as visualization in mathematics learning processes. Among the many technological tools used to support students' mathematics learning, GeoGebra is a leading dynamic mathematics software for learning and teaching mathematical proofs (Balacheff & de la Tour, 2019). Regarding the use of technology, students can be encouraged to evaluate existing software such as GeoGebra in the teaching and learning process to be used in the evaluation of proofs (Sümmermann et al., 2021).

Aim of Study

The aim of this study is to generalize by showing that the identity of the difference of two squares, which is expressed and proved in different ways for the square, which is itself a regular polygon, is valid for other regular polygons as well. In addition, the proofs are to observe the dynamic change of the visuals created depending on the n number of sides of the polygon with the GeoGebra program, which is a dynamic geometry software. The purpose of this proof is to deal with geometric thinking with the help of cognitive processes, as Duval (1998) stated.

METHOD

In this study, direct proof method was preferred based on the principle of conservation of measure (conservation of area), which is one of the basic principles of teaching geometry, namely "different geometric shapes with the same area can be created by dividing a geometric figure into smaller pieces according to some properties and combining the obtained pieces". Thus, with the direct proof method and GeoGebra, a dynamic geometry software, a new perspective was brought to the proof of the identity of the difference of two squares.

FINDINGS

Aslaner and Ilhan (2018) showed that an identity valid for right triangles, expressed for square and known as the Pythagorean relation, is also true for other regular polygons. Again, the correctness of the following proposition, which is an identity and expressed for the square, aroused curiosity.

Proposition 1

The proposition "for every real number a and b, $a^2-b^2=(a-b)(a+b)$" is known in the literature as the difference of two squares. The truth of this proposition is shown geometrically in **Figure 1**.

Proof

The difference of two square quadrilaterals ABCD and AEFG with side lengths a and b units is the polygon BCDGFE. This polygon is broken up by the line segment [CF], as shown in **Figure 1** and converted into a parallelogram or rectangle with the same area as the reflection, rotation, and translation movements known as geometrical transformations (**Figure 1**).

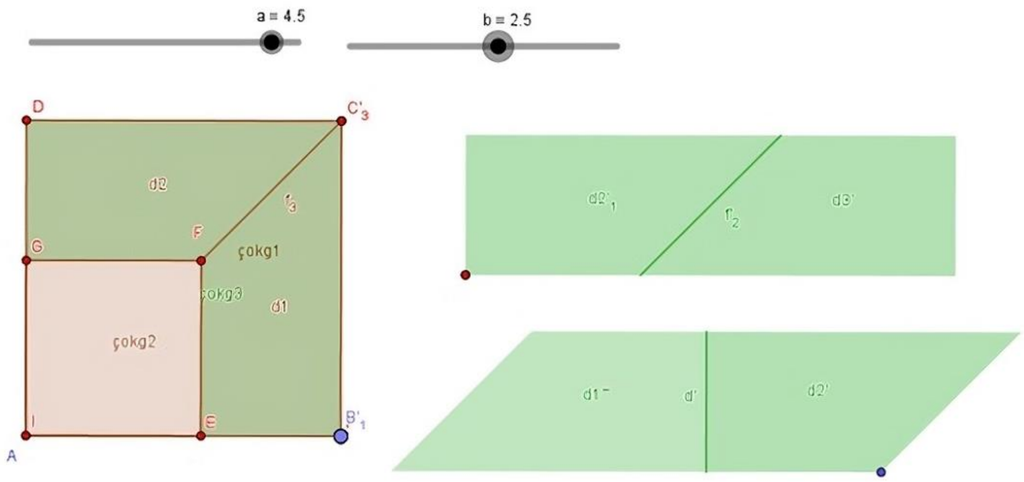

Figure 1. Difference of two squares (Source: Authors' own elaboration, using GeoGebra software)

This rectangle is a rectangle with side lengths (or parallelogram's side length and its height) (a-b) and (a+b), and its area is (a-b)(a+b). The difference of the areas of the quadrilaterals is a^2-b^2, and these two expressions are equal to each other. So the proposition given is true.

Now let's express this proposition for other regular polygons and show that it is true.

Proposition 2

The equation $a^2-b^2=(a-b)(a+b)$ is valid for all regular polygons with side lengths a and b (a >b).

The correctness of the proposition was tried to be proved in GeoGebra, a dynamic geometry software, with the help of visuals. In order to show that this proposition is true, firstly, it is expressed how to create an n-sided regular polygon with a side length in computer environment with dynamic software. For this, it is necessary to know the basic concepts and some properties of regular polygons. Polygons with congruent sides and angles are called ***regular polygons***. According to this definition, equilateral triangle and square are regular polygons.

Basic elements and properties of regular polygons:

- Every regular polygon has a **center point**.
- This point is the intersection point of the mid-perpendicular lines of the polygon.
- The polygon has a circle passing through its vertices.
- This circle is called the **circumcircle** of the polygon.
- Each regular polygon consists of n isosceles triangles, with the center point being the common vertex.
- The vertex angle of these triangles is called the **central angle** of the polygon.
- The measure of a central angle is $360^o/n$.

A regular polygon with n sides given an edge length is created in the dynamic software GeoGebra by applying the following steps.

- With the slider option for the number of sides, an integer variable n and a positive variable a for the **side length** are created.
- A line segment [AB]: f is drawn with the line segment *tool of given length*.

For the sides to be congruent,

- This line segment is projected to point B and the line segment f' is found.

For the measure of the exterior angle of the polygon to be drawn,

- An α angle is defined by typing $360^o/n$ on the **input** screen.

This angle is also the central angle of the polygon.

- The line segment f' is rotated around the point B by an angle of α and the line segment f'' is found.

This line segment is the second side of our polygon.

- The mid-perpendicular lines of these two sides are drawn and the intersection point is taken as the center point C.
- The line f''' is obtained by rotating the side f'' around C by α.

This line segment is the third side of our polygon.

If this process is repeated n-3 times, the desired polygon will be drawn. To do this, go to the **home** screen,

- List L1 is created by writing **array (rotate [f'', i α, C], i, 1, n-3)**, and a regular polygon with n sides is drawn (**Figure 2**).

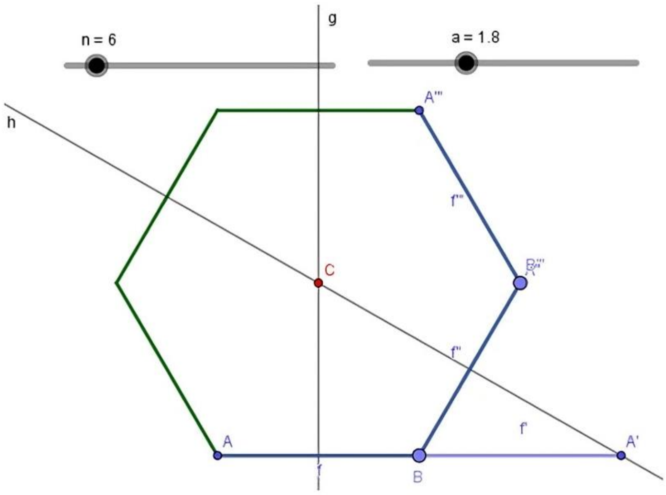

Figure 2. Formation of regular polygons (Source: Authors' own elaboration, using GeoGebra software)

In the window opened with the **create new tool** option from the **tools** menu of the program, the polygon (list 1) is transferred to the **output objects** area. **When next** is called, n and a numbers and point A are seen in the Input Objects field. A new tool symbol appears at the end of the program's menu list when a **name** (nSRP) is written instead of the tool name in the window that opens when it is called **next**. By clicking this tool, n and a are written respectively on the lines opened, and if a point is selected in an appropriate place on the screen, an n-sided regular polygon with a side length a unit will be drawn, starting at the selected point A. Here, by changing the number of sides of the polygon with the variable n, the size of the polygon with the variable a, and its location with the point A, a regular polygon with the desired size and number of sides is drawn in the desired environment. In the light of this information, the correctness of **proposition 2** can be shown, as follows, starting from the simplest regular polygon, the equilateral triangle;

for n=3;

The difference of triangles ABC and AED with side lengths a and b given below is the trapezoid BCDE (**Figure 3**).

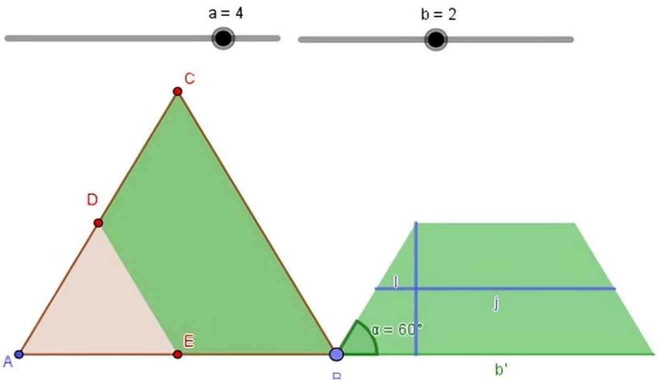

Figure 3. Difference region of two equilateral triangles (Source: Authors' own elaboration, using GeoGebra software)

The area of this trapezoid is the midsole multiplied by the height, $A = \frac{a+b}{2} \times (a-b)\sin 60 = \frac{\sqrt{3}}{4}(a-b)(a+b)$. Difference in the areas of the triangles is $\frac{\sqrt{3}}{4}a^2 - \frac{\sqrt{3}}{4}b^2 = \frac{\sqrt{3}}{4}(a^2 - b^2)$ and from the equality of these two expressions, for each real value of a and b; $a^2-b^2=(a-b)(a+b)$ obtained.

for n=5;

The difference of ABCDE and AFGHI pentagons with side lengths a and b units given below is the polygonal region FB … G given in **Figure 4**.

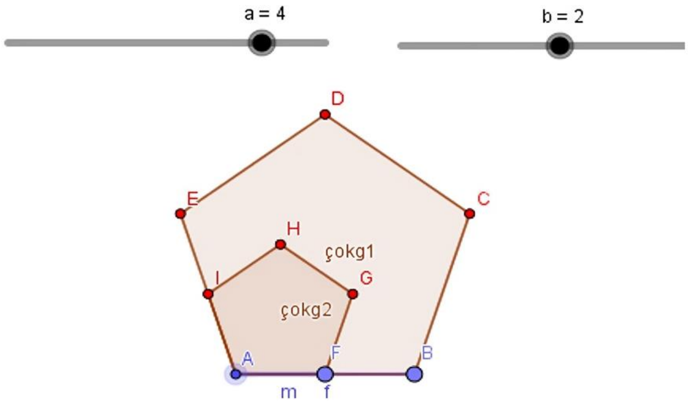

Figure 4. Area difference in regular pentagons (Source: Authors' own elaboration, using GeoGebra software)

It is difficult to break up this region as in a quadrilateral and turn it into a parallelogram or rectangle. Therefore, it is necessary to create a different shredding technique. Instead of having a common vertex and parallel sides as in **Figure 4**, polygons are placed so that their centers are common, and their sides are parallel.

For this, the following steps should be applied in the GeoGebra program.

The polygon to be created;

- For the number of sides, an integer variable n greater than three and a positive variable a for the side length are created with the **slider** tool.
- A line segment [AB]: f is drawn with the **line segment of given length** tool.
- This line segment has the middle perpendicular line and the midpoint C.

The center point of the polygon should be on this line. For the central angle of the polygon,

- An α angle is defined by typing 360°/n on the **input** screen.
- By drawing the line AB, this line is rotated around the point A by 90-α/2.
- The intersection point of these two lines is the center point of the D point polygon (**Figure 5**).

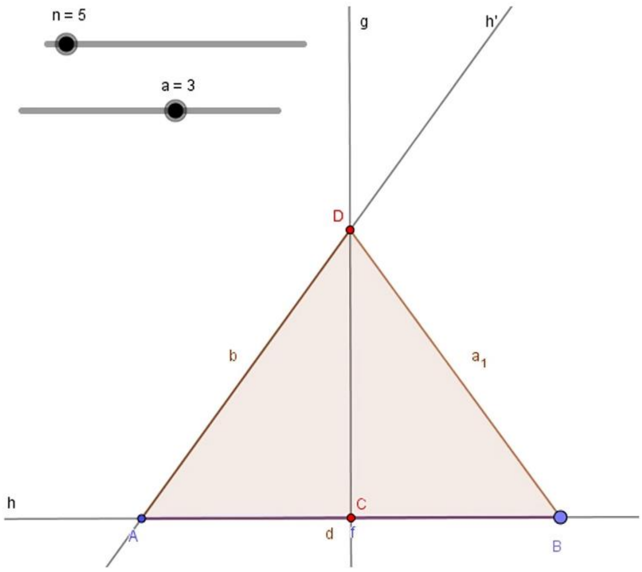

Figure 5. Isosceles triangle (Source: Authors' own elaboration, using GeoGebra software)

Triangle ABD is an isosceles triangle with apex α, base angles 90-α/2, side lengths $b = \frac{a}{2\sin\frac{\alpha}{2}}$ and base height $\frac{a}{2} = \cot\frac{\alpha}{2}$. So it has $s(ABD) = \frac{a^2}{4}\cot\frac{\alpha}{2}$ area.

- With the **regular polygon** tool, draw an n-sided polygon with one side [AB] (**Figure 6**).

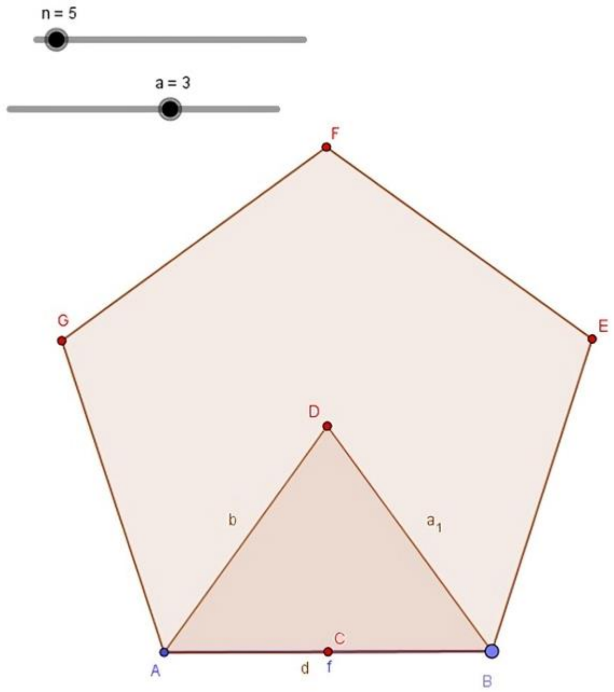

Figure 6. Formation of regular polygons (Source: Authors' own elaboration, using GeoGebra software)

This polygon consists of n isosceles triangles, with D being their common vertex.

- Other triangles can be drawn by rotating triangle ABD about D point (n-1) times. For this;
- **Array (rotate [ü2, i α, D], i, 1, n-1)** should be written on the **input** screen (**Figure 7**).

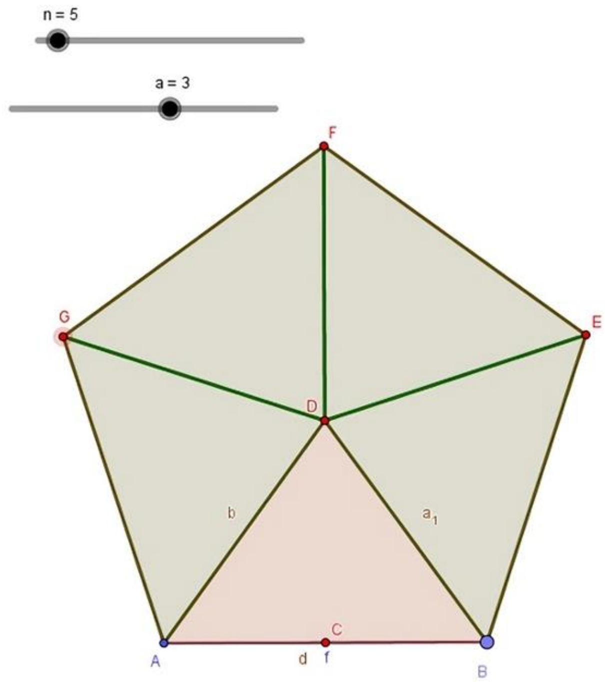

Figure 7. Segmented regular polygon (Source: Authors' own elaboration, using GeoGebra software)

The area of this polygon $A_1 = ns(ABD) = \frac{n}{4}\cot\frac{\alpha}{2}a^2$.

Let's construct a second polygon whose central point D has sides parallel to this polygon and one side length b units (0≤b≤a). For the side length of the second polygon,

- A variable b, ranging from 0 to a, is created with the **slider** tool.
- C (b/2) circle with center C and radius b/2 is drawn, with C being the midpoint of [AB] (**Figure 8**).
- From the point H, which is the intersection point of this circle with the line AB, the line perpendicular to AB is drawn.
- The intersection point of this point with [AD] is I, and the reflection of I' with respect to the middle perpendicular line is found I'.
- The polygon drawn by taking one side [II'] with the **regular polygon** tool is the second polygon (**Figure 8**).

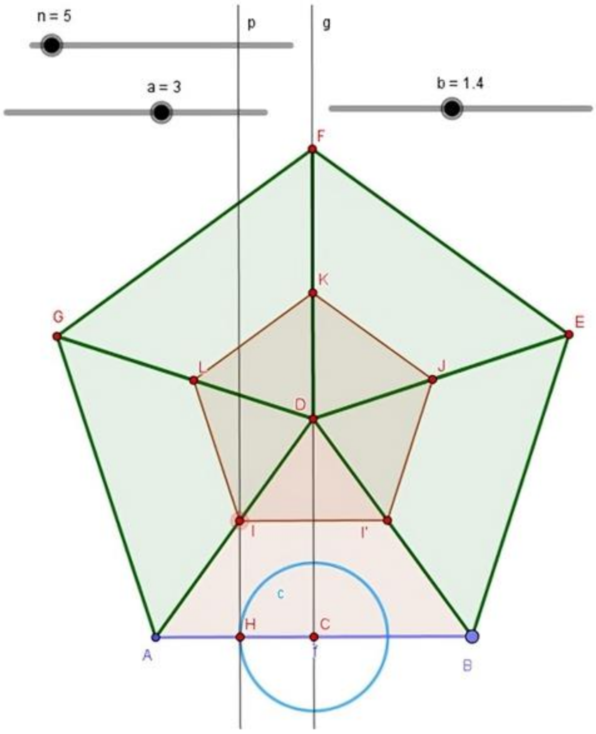

Figure 8. Second regular polygon (Source: Authors' own elaboration, using GeoGebra software)

Area of this polygon is $A_2 = \frac{n}{4}\cot\frac{\alpha}{2}b^2$ also and the area of the region that is the difference of the two polygons is, as follows:

$$A = A_1 - A_2 = \frac{n}{4}\cot\frac{\alpha}{2}(a^2 - b^2). \tag{1}$$

- Create trapezoid ABII' with the **Polygon** tool (**Figure 9**).

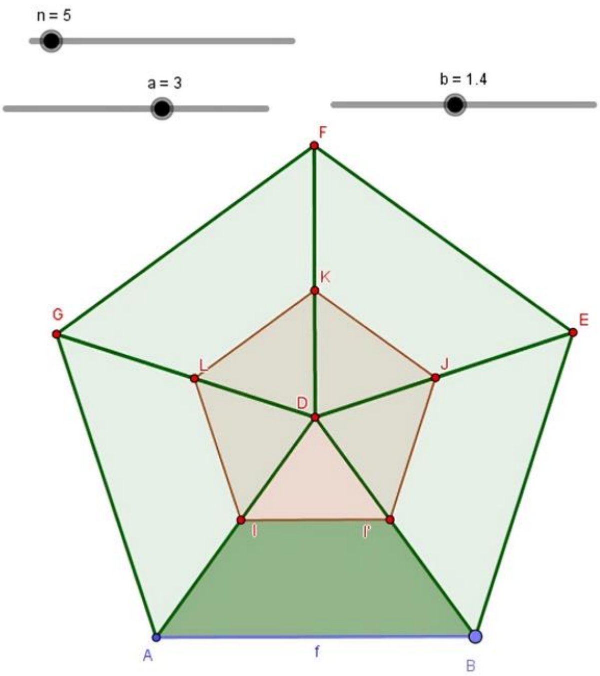

Figure 9. Trapezoid that creates a difference area (Source: Authors' own elaboration, using GeoGebra software)

This trapezoidal base lengths a and b are an isosceles trapezoid with height $h = \frac{a-b}{2}\cot\frac{\alpha}{2}$, the area of which is the midsole multiplied by the height, $A_y = \frac{1}{4}\cot\frac{\alpha}{2}(a-b)(a+b)$.

The region that is the difference of the two polygons consists of the union of n trapezoids. These trapezoids can be transformed into a trapezoid if n is odd, and a parallelogram if n is even, by applying reflection, rotation and translation movements, briefly called geometric transformations (**Figure 10**).

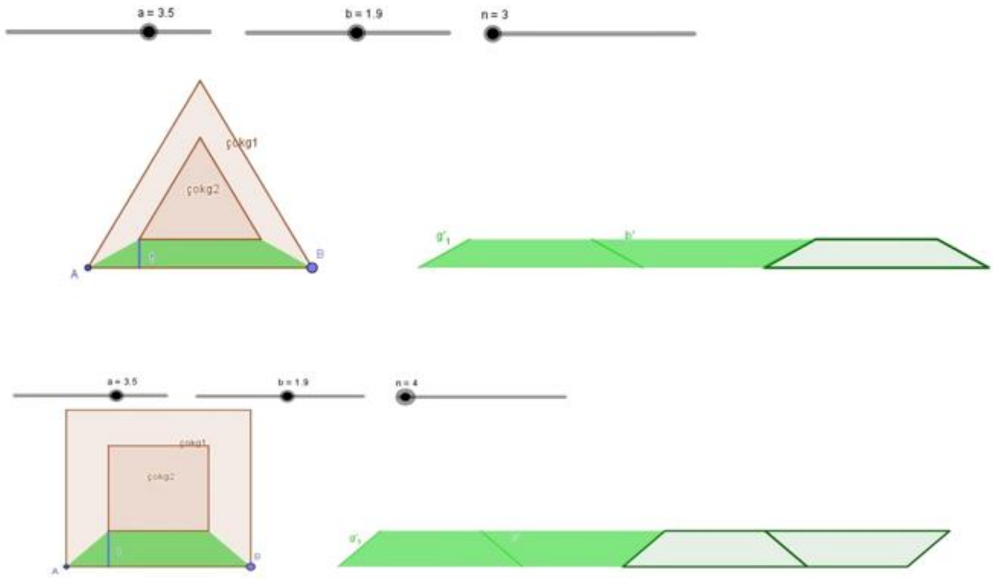

Figure 10. Area formed by combination of trapezoids (Source: Authors' own elaboration, using GeoGebra software)

The area of this trapezoid (or parallelogram):

$$A = nA_y = \frac{n}{4}\cot\frac{\alpha}{2}(a-b)(a+b). \quad (2)$$

The area of the difference region is equal. From these two equations, it can be said that the a^2-b^2=(a-b)(a+b) identity expressed for each real numbers a and b is valid for all regular polygons as well.

Depending on the selected edge length a in the geometric drawings of the proposition given in this illustration, the figure becomes quite large as the number of edges increases. To avoid this growth, the polygon can be confined to the circumcircle. In other words, the growth of the figure can be prevented by connecting the side length of the polygon to the radius of the circle. For this, a radius variable r, a variable dependent on the radius, and b a variable smaller than a; let us show that statement 2 is true.

For this, the polygon;

- An integer variable n, starting from three, is created for the number of sides.
- An α angle is defined by typing 360°/n on the **home** screen for the center angle.
- A positive variable r is created for the radius length.
- A circle A(r) with origin center and radius r is drawn (**Figure 11**).

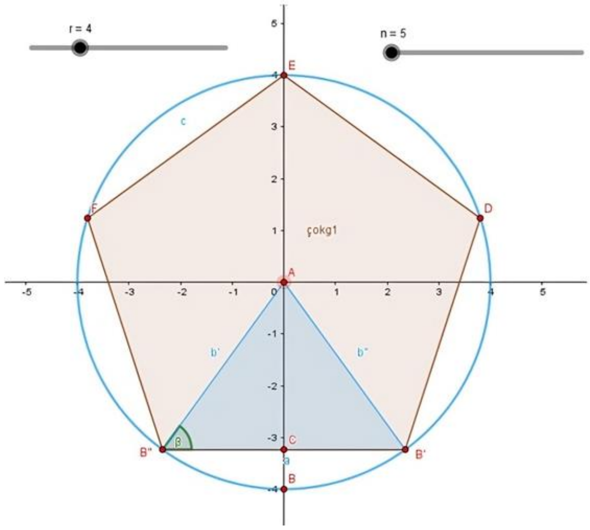

Figure 11. Formation of a regular polygon depends on a & r (Source: Authors' own elaboration, using GeoGebra software)

For the intersection point B of this circle with the y-axis, there are points B' and B''.

The base length of the isosceles triangle AB''B' depends on the radius from **Figure 10** $a = 2r\sin\frac{\alpha}{2}$.

- If the **regular polygon tool** is selected and the points B'' and B' are clicked, an n-sided regular polygon with one side length unit and the radius of its circumcircle is r will be drawn.

This regular polygon consists of n isosceles triangles with vertex center point, vertex angle α, base length, and side lengths r. To draw these triangles,

- **Array (rotate [ü1, i α], i, 1, n-1)** should be written on the **input** screen.
- A second polygon with a side length b is drawn by creating a variable b that varies between zero and a (**Figure 12**).

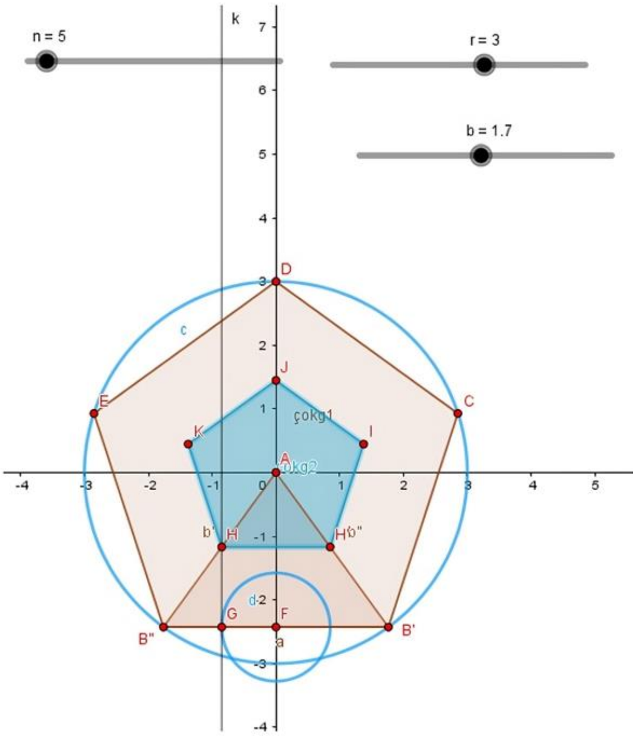

Figure 12. Regular polygon with b sides (Source: Authors' own elaboration, using GeoGebra software)

The difference of these two polygons consists of n trapezoids with base lengths a and b (**Figure 13**).

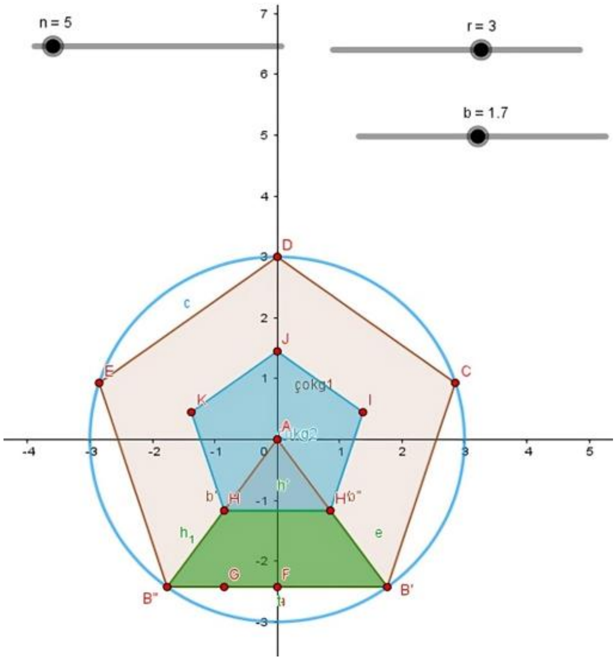

Figure 13. Difference area of two polygons (Source: Authors' own elaboration, using GeoGebra software)

When these trapezoids are combined in one place by applying reflection, rotation and translation movements, briefly known as geometric transformations, a trapezoid is obtained if n is odd, and a parallelogram if n is even (**Figure 14**).

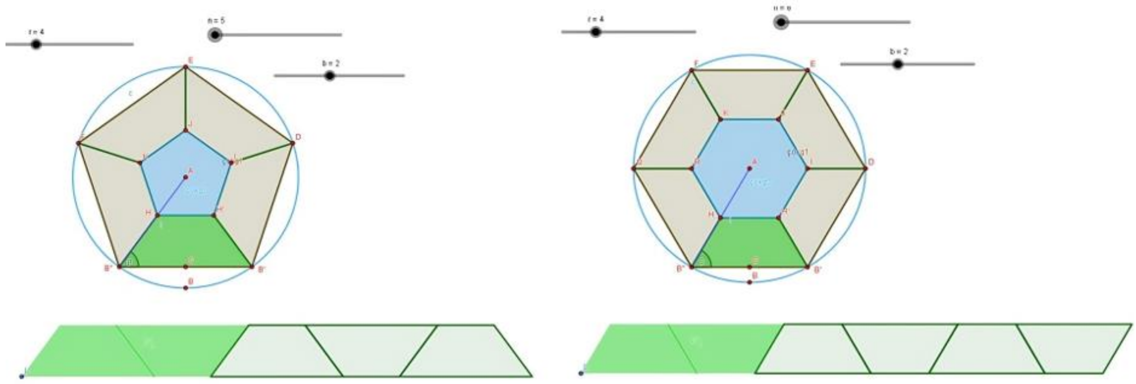

Figure 14. Whether n is odd or even (Source: Authors' own elaboration, using GeoGebra software)

The area of this trapezoid (or parallelogram) is equal to the difference of the areas of the two polygons. Therefore, it can be said that the two square difference identity expressed for the square is also true for other regular polygons. In regular polygons, as the number of sides increases, the polygon resembles a circle, and the polygonal region resembles a circle. For this reason, it is possible to use the formulas obtained by using sides and angles in polygons for circles without sides and corners and similar concepts. Accordingly, the same proposition can be expressed for the area of the annular region formed by taking two circle segments with the same center but different radii, instead of the trapezoid formed by taking the difference of two isosceles triangles with the same vertices and different side lengths (**Figure 15**).

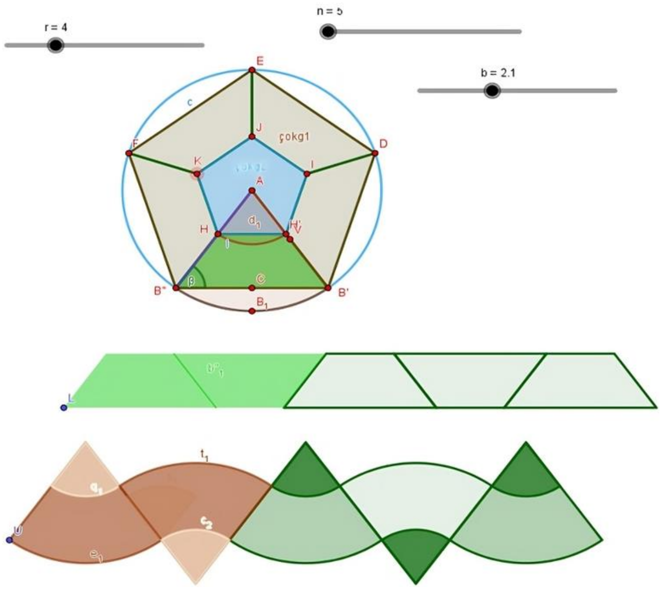

Figure 15. Transition from polygon to stripe space (Source: Authors' own elaboration, using GeoGebra software)

As n increases, the area of the trapezoid (or parallelogram) and the area of the strip converge. For n infinity, the difference is zero and the areas are equal to each other. Therefore, it can be said that the equation expressed and shown to be true for two regular polygons with different side lengths is also true for two circles with different radii. So before;

"Two equations with side lengths a and b (a>b), $a^2-b^2=(a-b)(a+b)$ are also valid for all regular polygons."

Proposition 2 expressed, as follows: "Equation $a^2-b^2=(a-b)(a+b)$ is valid for any two circles with radii a and b (a>b)." proposition can be brought to the literature.

DISCUSSION & CONCLUSIONS

As a result of enriching the teaching processes with dynamic software and digital materials, the proofs made with the help of visuals are becoming increasingly common day by day. The use of proofs in other learning areas, especially in mathematics education, has played an important role in the realization of meaningful learning. As a matter of fact, Hanna and Sidoli (2007) stated that visualization has positive contributions to the understanding of proofs. Visualization and proving in this way not only contributes to meaningful learning, but also helps to establish a cause-effect relationship. At this point, Malaty (2001) stated that visualization supports thinking based on cause-effect relationships and said that the ways of making mathematical meanings with visualization should also be taught. The basis of research on geometry, which is an important learning area of mathematics, lies in drawing and visualizing geometric concepts and creating generalizations based on these visuals (Kose, 2008). Hanna and Sidoli (2007) stated that the use of visualization in proof processes in mathematics education, where teaching with many geometric concepts is carried out, has a positive contribution to the understanding of proofs. Along with this study, there are many studies that talk about the role of visualization in mathematics teaching (Amrhein et al., 1997; Arcavi, 2003; Malaty, 2001; Tall, 2004).

Identities, which are frequently used in the field of learning algebra in mathematics, are sometimes proved with the help of geometric shapes and visuals. For the *difference of two squares*, which is one of these identities, two squares are drawn in general and visual proofs are made with the help of their areas. There are studies in the literature that use the classical visual proof of the difference of two squares calculated by square areas (Ozdemir et al., 2005; Yilmaz et al., 2016). However, it is crucial for meaningful learning to evaluate different drawing methods instead of using a single classical method while making these visual proofs. Ozdemir et al. (2005) stated that in the difference of two squares, it is necessary to first understand what the expression x^2-y^2 means geometrically. In this direction, in this study, the area relation expressed for the square difference identity of two squares is introduced and it is shown by the method of direct proof by creating different geometric shapes that it is also valid for other regular polygons and circle areas. The drawings given while making these proofs were created in the GeoGebra program. As a matter of fact, there are many recent studies in the literature stating that the GeoGebra program has significant contributions on visualization and proof in learning processes (Balacheff & de la Tour, 2019; Baye et al., 2021; Kohen et al. 2019; Sümmermann et al., 2021; Zhang et al., 2023; Ziatdinov & Valles, 2022). By developing a new fragmentation technique on the difference identity of two squares, correct results are obtained by using the formulas obtained for regular polygons such as triangle, square and pentagon, also for circles without sides and corners.

Limitations & Recommendations

The study is limited to the visual proof of two square difference identity using polygon areas in GeoGebra software. Within the framework of the results obtained in the research, the following suggestions can be made to researchers who want to work on this subject in the future:

1. In this research, a new visual proof is made for the difference of two squares, which is a type of identity. New proofs can be developed for other similar identities as well.
2. Considering the contribution of visualization to meaningful learning, such proofs can be integrated into teaching processes with GeoGebra software.
3. Studies that include new technological proofs can be done with augmented and virtual reality applications.
4. The effect of dynamic geometry software such as GeoGebra on the development of such proofs can be evaluated through action or experimental research.
5. The contribution level of meaningful dynamic software to learning can be determined by creating teaching averages in which such applications are made.

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REFERENCES

Alsina, C., & Nelsen, R. B. (2006). *Math made visual creating images for understanding mathematics*. The Mathematical Association of America. https://doi.org/10.5948/UPO9781614441007

Amrhein, B., Gloor, O., & Maede, R., (1997). Visualizations for mathematics courses based on a computer algebra system. *Journal of Symbolic Computation, 23*(1), 447-452. https://doi.org/10.1006/jsco.1996.0099

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*(1), 215-241. https://doi.org/10.1023/A:1024312321077

Aslaner, R., & Ilhan, A. (2018). Pythagoras connection expressed for square application of other plain polygons and appliances. *The Journal of Buca Faculty of Education, 45*(1), 55-67.

Balacheff, N., & de la Tour, T. B. (2019). Proof technology and learning in mathematics: Common issues and perspectives. In G. Hanna, D. A. Reid, & M. de Villiers (Eds.), *Proof technology in mathematics research and teaching* (pp. 349-365). Springer. https://doi.org/10.1007/978-3-030-28483-1_17

Baye, M. G., Ayele, M. A., & Wondimuneh, T. E. (2021). Implementing GeoGebra integrated with multi-teaching approaches guided by the APOS theory to enhance students' conceptual understanding of limit in Ethiopian universities. *Heliyon, 7*(5), e07012. https://doi.org/10.1016/j.heliyon.2021.e07012

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modeling process. *ZDM-Mathematics Education, 38*(2), 86-95. https://doi.org/10.1007/BF02655883

Chan, K. K., & Leung, S. W. (2014). Dynamic geometry software improves mathematical achievement: Systematic review and meta-analysis. *Journal of Educational Computing Research, 51*(3), 311-325. https://doi.org/10.2190/EC.51.3.c

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana, & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century: An ICMI study* (pp.37-52). Kluwer.

English, L. D., & Watters, J. J. (2005). Mathematical modeling in third-grade classrooms. *Mathematics Education Research Journal, 16*(3), 59-80. https://doi.org/10.1007/BF03217401

Granberg, C., & Olsson, J. (2015). ICT-supported problem solving and collaborative creative reasoning: Exploring linear functions using dynamic mathematics software. *The Journal of Mathematical Behavior, 37*(1), 48-62. https://doi.org/10.1016/j.jmathb.2014.11.001

Hamersma, P. S. (2002). *Effects of using a computer-based visualization tool to learn complex numbers in trigonometry* [PhD dissertation, University of South Florida].

Hanna, G., & Sidoli, N. (2007). Visualization and proof: A brief survey of philosophical perspectives. *ZDM- Mathematics Education, 39*(1), 73-78. https://doi.org/10.1007/s11858-006-0005-0

Hanna, G., & Williers, M. (2008). ICMI study 19: Proof and proving in mathematics education. *ZDM-Mathematics Education, 40*(1), 329-336. https://doi.org/10.1007/s11858-008-0073-4

Johnson, H. L. (2022). Task design for graphs: Rethink multiple representations with variation theory. *Mathematical Thinking and Learning, 24*(2), 91-98. https://doi.org/10.1080/10986065.2020.1824056

Kohen, Z., Amram, M., Dagan, M., & Miranda, T. (2019). Self-efficacy and problem-solving skills in mathematics: The effect of instruction-based dynamic versus static visualization. *Interactive Learning Environments, 30*(4), 759-778. https://doi.org/10.1080/10494820.2019.1683588

Kose, N. Y. (2008). *Determining fifth grade primary school students? Understanding of symmetry using dynamic geometry software Cabri geometry: An action research* [Doctoral thesis, Anadolu University].

Liang, H. N., & Sedig, K. (2010). Can interactive visualization tools engage and support pre-university students in exploring non-trivial mathematical concepts? *Computers & Education, 54*(1), 972-991. https://doi.org/10.1016/j.compedu.2009.10.001

Malaty, G. (2001). *The role of visualization in mathematics education: Can visualization promote the causal thinking?* http://woulibrary.wou.edu.my/weko/eed502/the_role_of_visualisation_in_math_edu.pdf

NCTM. (2020). *Principles and standards for school mathematics*. National Council of Teachers of Mathematics.

Ozdemir, M. E., Duru, A., & Akgun, L. (2005). Two or three dimensional thinking: Visualization of some identities with two and three dimensional geometrical figures. *Kastamonu Education Journal, 13*(2), 527-540.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics. In A. Gutierrez, & P. Borero (Eds.), *Handbook of research on psychology of mathematics education: Past, present and future* (pp. 205-235). Sense Publishers. https://doi.org/10.1163/9789087901127_009

Putri, R. I., & Santosa, R. H. (2015). The effectiveness of REACT strategy viewed from learning achievement, problem solving ability, mathematical connection, self-efficacy. *Jurnal Riset Pendidikan Matematika* [*Journal of Mathematics Education Research*]*, 2*(2), 262-272. https://doi.org/10.21831/jrpm.v2i2.7345

Rahim, M. H., & Siddo, R. (2009). The use of visualization in learning and teaching mathematics. In A. Rogerson (Ed.), *Proceedings of the 10th International Conference: Models in Developing Mathematics Education*.

Rösken, B., & Rolka, K. (2006). A picture is worth a 1000 words–the role of visualization in Mathematics learning. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Ed.), *Proceedings of the 30th Conference of the International Group for Psychology of Mathematics Education* (pp. 457464). PME.

Stylianou, D. A., & Silver, E. A. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Journal of Mathematical Thinking and Learning, 6*(4), 353-387. https://doi.org/10.1207/s15327833mtl0604_1

Sümmermann, M. L., Sommerhoff, D., & Rott, B. (2021). Mathematics in the digital age: The case of simulation-based proofs. *International Journal of Research in Undergraduate Mathematics Education, 7*(3), 438-465. https://doi.org/10.1007/s40753-020-00125-6

Tall, D. (2004). *A versatile theory of visualization and symbolisation in mathematics* [Plenary presentation]. The International Commission for the Study and Improvement of Mathematics Teaching.

Yilmaz, A., Gokkurt, B., & Usta, N. (2016). Teachers' opinions on teaching the logical inference of identities and using skills of these logical inferences. *Bartin University Journal of Faculty of Education, 5*(1), 106-127. https://doi.org/10.14686/buefad.v5i1.5000172655

Yulianto, A. R., Rochmad, R., & Dwidayati, N. K. (2019). The effectiveness of core models with scaffolding to improve the mathematical connection skill. *Journal of Primary Education, 8*(4), 1-7.

Zhang, Y., Wang, P., Jia, W., Zhang, A., & Chen G. (2023). Dynamic visualization by GeoGebra for mathematics learning: A metaanalysis of 20 years of research, *Journal of Research on Technology in Education, 2023*(August), 1-23. https://doi.org/10.1080/15391523.2023.2250886

Ziatdinov, R., & Valles, J. R. (2022). Synthesis of modeling, visualization, and programming in GeoGebra as an effective approach for teaching and learning STEM topics. *Mathematics, 10*(3), 398. https://doi.org/10.3390/math10030398

Zulnaidi, H., Oktavika, E., & Hidayat, R. (2020). Effect of use of GeoGebra on achievement of high school mathematics students. *Education and Information Technologies, 25*(1), 51-72. https://doi.org/10.1007/s10639-019-09899-y