

From process to object: Scaffolding reification and proceptual fluency in infinite series with accessible dynamic visualizations

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ABSTRACT

Transforming an infinite process into a mathematical object is a core cognitive challenge in calculus learning, yet many classrooms lack access to specialized dynamic visualization software. This study explores whether intentionally designed animations in Microsoft PowerPoint, a widely available tool with basic animation capabilities, can scaffold this cognitive leap within the Vietnamese context. Using an explanatory sequential mixed-methods design, 60 eleventh-grade students were assigned to either an experimental group learning with theory-guided PowerPoint Dynamic Multiple Representations (DMRs) or a control group receiving traditional instruction over a two-week unit on infinite geometric series. The experimental group demonstrated significantly higher conceptual gains ($d = 2.25$, a large effect comparable to intensive intervention studies in mathematics education) and lower cognitive load ($d = 1.52$). Qualitative analysis revealed that the animations triggered cognitive mechanisms of *completion* and *totality* construction, enabling students to view infinite sums as manipulable objects rather than unending processes. This study introduces a “high-design, low-barrier” model that promotes educational equity by enabling sophisticated mathematics instruction without specialized software. This paper offers practical design principles that teachers can implement immediately using existing classroom technology.

Keywords: infinite geometric series, reification, procept, totality, dynamic multiple representations

INTRODUCTION

The transition from algebraic thinking, characterized by finiteness and discreteness, to analytical thinking, with its infinite and continuous nature, represents a significant cognitive barrier for high school students (Tall, 1992). Within this context, the concept of infinite series poses particular difficulties. Studies indicate that students often view infinity primarily as an ongoing process that never concludes (potential infinity), rather than recognizing it as a static, completed mathematical entity (actual infinity) (Dubinsky et al., 2005; Fischbein, 2001). This entrenched process perspective hinders students from flexibly manipulating infinite sums as finite mathematical objects (Brown et al., 2010).

To explain this difficulty, Reification theory posits that the cognitive transition from a process to an object is a challenging for many students to achieve without scaffolding (Sfard, 1991). Simultaneously, Procept theory emphasizes the importance of fluency in using the same mathematical symbol (such as the sum symbol S) to represent both an operational process and a static concept (Gray & Tall, 1994). More recently, extended APOS theory has identified a crucial intermediate stage termed Totality, situated between Process and Object (Dubinsky et al., 2013). At this stage, learners must envision the infinite process as a mentally complete whole before they can apply actions to it to form a manipulable mathematical object (Villabona et al., 2022; Weller et al., 2009).

Dynamic Multiple Representations (DMRs) are regarded as a powerful tool to support students in grasping abstract concepts by combining visual, symbolic, and linguistic forms (Ainsworth, 2006; Bautista et al., 2023). Unlike static representations, dynamic representations offer distinct advantages in illustrating continuous change and spatiotemporal processes (Finzer & Jackiw, 1998; Ploetzner & Lowe, 2004). The flexible coordination of different registers of representation, visual, symbolic, and linguistic, is central to mathematical cognitive activity (Duval, 2006; Kaput, 1992). However, specialized dynamic mathematics software (such as GeoGebra or Mathematica) can create access barriers in classrooms lacking financial resources or technological proficiency (Becta, 2004). Moreover, if not carefully designed, integrating multiple representations simultaneously can induce cognitive load and the split-attention effect for learners (Chandler & Sweller, 1992; Lowe, 2003).

Recent research indicates that Microsoft PowerPoint, a nearly ubiquitous presentation software globally, has the potential to address this access issue (Nam et al., 2024). While PowerPoint is one of the most common presentation tools globally, advanced animation features such as Morph are only available in recent versions (PowerPoint 2019 and Microsoft 365). By flexibly leveraging dynamic features (such as the Morph transition), teachers can create visual models of the sum of an infinite geometric series, helping students construct a geometric representation of the limit and reducing the abstraction of calculus. When designed with clear pedagogical intentionality, PowerPoint animations not only attract attention but also synchronize representational aspects, supporting students in recognizing the convergence process of an infinite series (Hashemi et al., 2012).

Although the theories of reification, procept, and totality are frequently employed as post-hoc analytical lenses for describing how students learn, their potential for integration into a proactive pedagogical design framework remains underexploited. A gap persists between cognitive theory and instructional design: these frameworks have rarely been operationalized as *proactive blueprints* for creating instructional materials that actively guide learners through the process-to-object transition (Asiala et al., 1996; Dubinsky, 1991). While prior studies have applied APOS theory to design instruction for specific mathematical topics (Asiala et al., 1996), these efforts typically focused on a single theoretical lens and treated the Totality stage primarily as an analytical category rather than an explicit design target. Moreover, they often relied on specialized software, limiting accessibility. Our study addresses these gaps by (1) synthesizing Reification, Procept, and extended APOS (Totality) into a unified RPT design framework, (2) explicitly operationalizing the Totality stage as a pedagogical objective with specific visual scaffolds, and (3) demonstrating that this theory-driven design can be implemented with widely available technology.

Using an explanatory sequential mixed-methods research design (QUAN → QUAL) with 60 Grade 11 students (divided into an experimental group learning with the RPT-guided DMRs and a control group receiving traditional instruction), this study pursues two main objectives. Theoretically, it transforms Reification, Procept, and Totality theories from passive analytical tools into a generative framework for instructional material design. Practically, it demonstrates that pedagogical intentionality grounded in theoretical foundations, rather than technological complexity, is the key to deep learning, thereby suggests a promising model for equitable, high-quality instruction that merits further testing across diverse educational contexts.

Beyond cognitive considerations, the findings are also interpreted through Cultural-Historical Activity Theory (CHAT; Engeström, 2015), which views learning as mediated by cultural tools and social contexts. This lens helps us examine how the use of ubiquitous technology might reshape classroom activity and address structural inequities in access to dynamic visualization.

This study addresses two research questions (RQs):

RQ1: To what extent do RPT-guided Dynamic Multiple Representations (DMRs) in PowerPoint produce greater conceptual understanding gains and lower cognitive load compared to traditional instruction for learning infinite geometric series?

RQ2: What cognitive mechanisms explain how RPT-guided DMRs facilitate the process-to-object transition in learners' understanding of infinite series?

MATERIALS AND METHODS

An explanatory sequential mixed-methods design (QUAN → QUAL) was adopted (Creswell & Plano Clark, 2017). The primary quantitative phase utilized a quasi-experimental design to assess the efficacy of the theory-driven intervention. The subsequent qualitative phase was then conducted to explain and elaborate on the quantitative findings, providing a rich, theory-grounded understanding of the underlying cognitive mechanisms. This integration allowed for a comprehensive evaluation of both the *outcomes* and the *processes* of learning facilitated by the integrated Reification-Procept-Totality (RPT) framework.

Research Design and Participants

This study employed a stratified randomized controlled trial (RCT) design with pretest-posttest control group structure. Students were first stratified based on prior mathematics achievement (first-semester grades) and then randomly assigned (using a computer-generated random number sequence) to either the experimental group ($n = 30$) or the control group ($n = 30$). While the study was conducted in intact classrooms—precluding fully independent random selection from the broader population—the stratified random assignment at the student level within each classroom meets the methodological criteria for a randomized experiment (Shadish et al., 2002). It is acknowledged that the lack of random selection from the population limits generalizability, but the internal validity of group equivalence is ensured by the randomization procedure.

A quasi-experimental pretest-posttest control group design was implemented over a two-week instructional unit. Participants were 60 eleventh-grade students (aged 16–17 years) from two urban public high schools in Vietnam.

The two urban public high schools are located in Ho Chi Minh, Vietnam, and serve mixed socioeconomic populations (ranging from lower-middle to upper-middle income). Both schools follow the national Vietnamese mathematics curriculum, and neither had prior experience with the instructional approach used in this study. Students had no previous formal instruction on infinite series. The same experienced mathematics teacher (with 15 years of teaching experience) delivered all lessons in both groups using detailed lesson scripts, ensuring consistency. In both schools, computer labs are available but primarily used for computer science classes; students had basic familiarity with PowerPoint from prior presentations but not with dynamic animations in mathematics. The intervention was conducted in regular mathematics classrooms with a single projector and screen, which is typical for these schools.

To ensure group equivalence, a stratified random assignment procedure was used. Students were first stratified based on their prior mathematics achievement (first-semester grades), then randomly assigned (using a computer-generated number sequence)

Table 1. Participant demographics and baseline equivalence

Group	N	Male	Female	Mean Age (SD)	Prior Math Score, M (SD)
Experimental	30	14	16	16.8 (0.4)	7.9 (0.7)
Control	30	16	14	17.1 (0.5)	7.8 (0.6)

Table 2. Theoretical principles, design features, and expected outcomes in the RPT-guided DMR intervention

Theoretical Principles	Specific PowerPoint Design Features	Expected Cognitive Outcomes
Visual Scaffolding for Completion and Totality	Morph transition to progressively fill a unit right triangle; deliberate 1.5-second visual pause after completion; explicit textual cue (“The process is mentally complete, the sum exists as a whole”).	Construction of a stable mental <i>Totality</i> ; recognition of the infinite sum as a completed whole.
Sequencing Actions to Build Facets		
<i>Facet 1 (Recognition)</i>	After completion, the filled rectangle labeled “S” and enclosed in set notation {S}; on-screen text: “The sum S is a complete entity, a real number: $S \in \mathbb{R}$.”	Solidification of the sum as a named, static object within a mathematical set.
<i>Facet 2 (Comparison)</i>	Parallel animations for series with $r = \frac{1}{2}$ and $r = \frac{1}{3}$; prompt: “Which final sum, S_1 or S_2 , is larger? Why?”	Holistic comparison of sums as whole objects based on visual properties rather than procedural computation.
<i>Facet 3 (Operational Manipulation)</i>	After establishing a value for S (e.g., $S = 2$), new equation “ $3 \times S = ?$ ” appears; three copies of the filled rectangle merge to model multiplication; symbolic task: “If $S = \frac{a}{1-r} = 4$ and $r = 0.5$, find the first term a .”	Treatment of S as a manipulable algebraic variable; proceptual fluency with symbolic notation.
Triadic Coordination (Visual ↔ Symbolic ↔ Verbal)	Synchronized display of visual filling, symbolic formula, and verbal cues throughout the animations.	Continuous reinforcement of the dual nature of S as both <i>process</i> and <i>object</i> ; reduced split-attention effect.

to either the experimental group ($n = 30$) or the control group ($n = 30$). Thus, this study employs a stratified randomized controlled trial (RCT) design with a pretest-posttest control group structure. Although the study was conducted in intact classrooms—which prevents fully independent random selection from the broader population—the within-classroom student-level random assignment satisfies the methodological criteria for a randomized experiment (Shadish et al., 2002). While acknowledging that the lack of random selection limits external generalizability, but internal validity is ensured by the randomization procedure. The experimental group received instruction via theory-driven PowerPoint DMRs, while the control group received traditional lecture-based instruction with static diagrams and boardwork. Independent-samples t-tests confirmed no significant baseline differences between groups in prior math scores ($p = .54$) or age ($p = .21$) (see **Table 1**). Ethical approval was obtained from the university IRB, and informed assent/consent were secured from all students and their guardians.

Researcher Positionality: The first author and the second author are mathematics educators with 20 years of classroom experience, which informed the design of the PowerPoint DMRs. To minimize bias, the instructor delivering the intervention was not involved in test development or qualitative analysis. All interview coding was conducted by two independent researchers, and member checking was performed with six participants.

Intervention: Theory-Driven DMRs Design and Implementation

Both groups completed six standardized 45-minute lessons on infinite geometric series, covering convergence, the sum formula $S = \frac{a}{1-r}$, and conceptual applications. The same experienced instructor, using detailed lesson scripts, taught all sessions to control for teacher effects. Instructional fidelity was monitored via direct observation of 20% of randomly selected sessions using a checklist, with 95% adherence to the protocol.

The core distinction was the instructional medium and its design principles. The control group used static textbook images and boardwork. The experimental intervention was a theory-driven Dynamic Multiple Representation (DMR) package designed in Microsoft PowerPoint, which operationalized the Integrated RPT framework through two core design pillars: visual scaffolding for completion and totality construction, and sequenced actions to build multiple facets of the object conception. **Table 2** summarizes how each theoretical principle was translated into specific PowerPoint design features and the expected cognitive outcomes.

The DMRs were designed to guide learners through a visual narrative that helps resolve the infinite process into a totality. The Morph transition created seamless animations of the filling process for the canonical series $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^n + \dots$. **Figure 1** shows a key frame from this animation,, where a right triangle partitioned into progressively smaller similar triangles is progressively filled to represent the partial sums converging to the total area of 2.

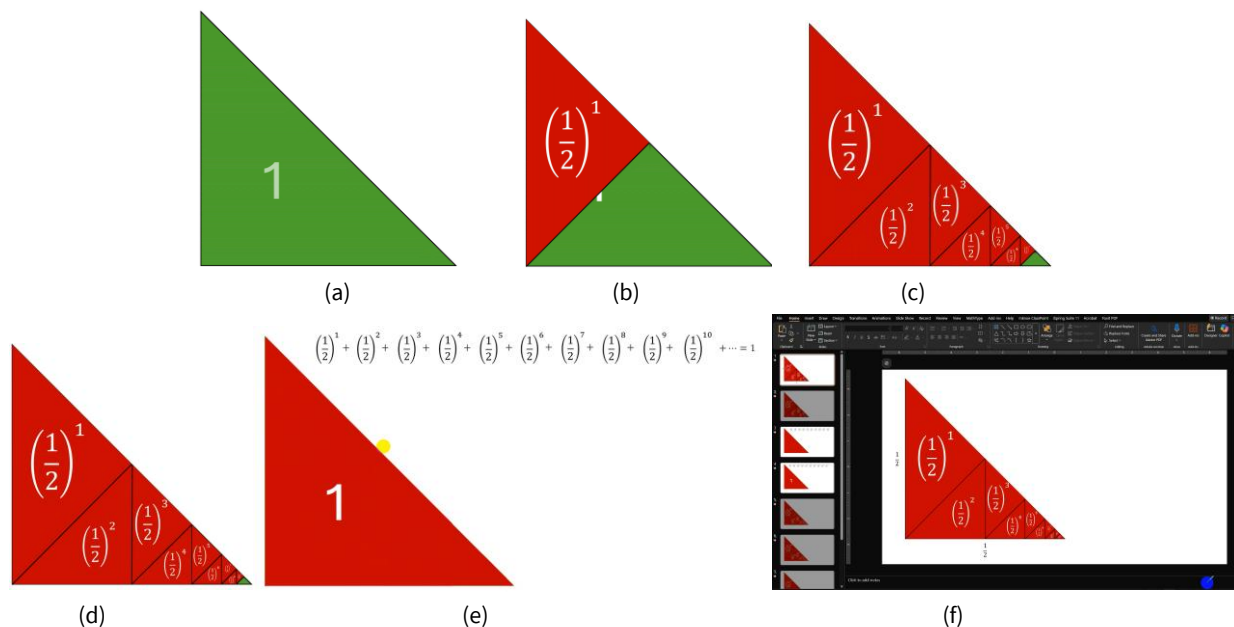


Figure 1. Key frames from the PowerPoint Morph animation illustrating the convergence of $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^n + \dots$ using a right triangle partitioned into progressively smaller similar triangles (Source: Authors' own elaboration)

After the final frame, the 1.5-second pause and explicit cue prompted students to mentally “complete” the infinite process. Building on this totality, the sequenced actions systematically constructed the three facets of an object conception: recognition (identifying S as an entity), comparison (evaluating sums holistically), and operational manipulation (using S in algebraic equations). For the comparison facet, parallel animations illustrated series with different common ratios, such as $r = \frac{1}{2}$ and $r = \frac{1}{3}$, as well as a generalized representation using the formula $S = a + ar + ar^2 + \dots = \frac{a}{1-r}$.

Instruments and Validation

Quantitative instruments

Conceptual Understanding Test: A 20-item researcher-designed test measured three sub-constructs: (a) *Process Understanding* (computation), (b) *Object Conception* (interpreting the infinite sum as an object), and (c) *Representational Fluency* (translating between visual, symbolic, verbal modes). The test included items specifically designed to probe the different facets of object conception. For example:

Facet 1 (Recognition): ‘Which statement best defines the sum of a convergent infinite geometric series?’ (Options distinguishing between limit-as-object vs. unending process).

Facet 2 (Comparison): ‘Given two convergent series with the same common ratio but different first terms, explain which sum is larger without calculating the exact value.’

Facet 3 (Manipulation): ‘If the sum S of a series is known to be 5, and it is used as the first term in a new series with ratio 0.2, what is the sum of the new series?’

The Conceptual Understanding Test underwent a two-stage validation process. First, three mathematics education researchers with expertise in cognition and assessment reviewed the items for content validity and alignment with the RPT framework. Based on their feedback, three items were revised and one was replaced. Second, the instrument was pilot-tested with 30 eleventh-grade students from a similar demographic (not part of the main study) to establish psychometric properties. Cronbach’s α coefficients were: Process Understanding (.82), Object Conception (.78), and Representational Fluency (.80), indicating acceptable internal consistency. Item discrimination indices ranged from .35 to .62, suggesting adequate item quality.

Cognitive Load Scale: An adapted NASA-TLX (Hart & Staveland, 1988) measured perceived cognitive load on a 9-point Likert scale across six subscales (e.g., Mental Demand, Frustration). Its adaptation for the mathematics classroom context was validated in a prior pilot study. The NASA-TLX scale had a Cronbach’s α of .86 in our sample.

Qualitative instruments

Semi-Structured Interview Protocol: A purposive sample of 12 students (6 per group, stratified by high/medium/low pretest scores) participated in post-intervention interviews (25–40 minutes). The protocol was explicitly designed to probe the theoretical constructs (e.g., “Can an infinite sum be ‘finished’? What in the animation supports your view?”, “Can you use the symbol S in an equation like a number?”).

The qualitative interview protocol was developed based on the RPT framework and was piloted with two students (not in the main sample) to check clarity and length. Inter-coder reliability for the thematic analysis was Cohen’s $\kappa = .87$, based on 25% of the transcripts coded independently by two researchers.

Table 3. Pre-test and post-test descriptive statistics for conceptual understanding

Group	N	Pre-test, M (SD)	Post-test, M (SD)	Mean Gain (SD)
Experimental	30	12.4 (2.1)	18.7 (1.8)	6.3 (1.5)
Control	30	12.2 (2.3)	14.9 (2.0)	2.7 (1.6)

Data Collection and Analysis Procedure

Data collection

The study followed a sequential data collection procedure over three weeks. In Week 1, all students completed the Conceptual Understanding pre-test. During Weeks 1 and 2, the six-lesson intervention was implemented for both groups. At the end of Week 2, students completed the post-test and the NASA-TLX cognitive load survey. In Week 3, in-depth semi-structured interviews were conducted with the purposively selected 12 students (6 per group). Each interview lasted 25–40 minutes and was audio-recorded for later transcription.

Data analysis

Quantitative analysis: Pre-test scores were used as a covariate. A one-way ANCOVA was conducted with post-test score as the dependent variable and group (experimental vs. control) as the independent variable. Subscale scores (Process Understanding, Object Conception, Representational Fluency) were analyzed using a repeated-measures ANOVA with Group as the between-subjects factor and Time (pre-test, post-test) as the within-subjects factor. Independent samples t-tests compared cognitive load scores between groups. To test the hypothesized mediation mechanism, a mediation analysis was performed using Hayes' PROCESS Macro (Model 4), with instructional group as the predictor, cognitive load as the mediator, and post-test score as the outcome. All statistical assumptions (normality, homogeneity of variance) were checked and met.

Qualitative analysis: Interview transcripts were analyzed using Framework Analysis. A deductive coding scheme was derived from the RPT framework, including codes for *completion*, *totality*, *facet 1 (recognition)*, *facet 2 (comparison)*, *facet 3 (operational manipulation)*, *procept-as-process*, and *procept-as-object*. Two researchers independently coded a subset of transcripts (25%) to establish inter-coder reliability, achieving Cohen's $\kappa = .87$. Discrepancies were resolved through discussion. The remaining transcripts were coded by the first researcher using the finalized scheme.

Integration of quantitative and qualitative findings: Following the explanatory sequential design, qualitative results were used to explain and elaborate on the quantitative findings. A joint display table (presented in Section *Alignment with Reification, Procept, and Totality Frameworks*) was constructed to map specific quantitative outcomes onto the cognitive mechanisms identified in the qualitative analysis, providing a coherent, integrated interpretation.

Ensuring Rigor and Validity

Multiple strategies were employed to ensure trustworthiness:

Construct Validity: Instruments were developed and validated through expert review and pilot testing.

Internal Validity: Use of a control group, randomization, ANCOVA to control for prior ability, and lesson scripting to ensure fidelity.

Theoretical Validity: The qualitative analysis was guided by and sought evidence for the *a priori* theoretical framework (RPT).

Integration Validity: The sequential mixed-methods design and the use of a joint display allowed for genuine triangulation and explanatory insight.

Replicability: The DMRs design principles are described in sufficient detail, and sample slides are provided in supplementary materials to allow for replication and adaptation.

To enhance trustworthiness, our qualitative analysis included a deliberate search for disconfirming evidence (negative cases) in both groups. These cases are reported in Section *Counter-Examples and Negative Cases*.

RESULTS

This section presents findings from the explanatory sequential mixed-methods analysis. Quantitative results on learning outcomes and cognitive load are first reported, followed by qualitative evidence that elucidates the underlying cognitive mechanisms of completion, totality formation, and facet development. Finally, an integrated synthesis is presented through a joint display.

Quantitative Results

Preliminary analyses and descriptive statistics

Descriptive statistics for the conceptual understanding pre-test and post-test scores are presented in **Table 3**. Both groups showed improvement from pre-test to post-test, but the experimental (DMRs) group demonstrated substantially greater gains.

All assumptions for parametric analysis were met. Data were normally distributed (Shapiro-Wilk $p > .05$ for all), and homogeneity of variance was confirmed (Levene's test, $p = .214$). The pre-test score was a significant covariate, $F(1, 57) = 38.45$, $p < .001$, $\eta^2 = 0.40$, justifying the use of ANCOVA.

Table 4. Pre-test and post-test scores by conceptual sub-construct

Construct	Group	Pre-test, M (SD)	Post-test, M (SD)	F (Interaction)	Partial η^2
Process Understanding	Experimental	4.2 (1.1)	5.7 (0.9)	18.42	.24
	Control	4.1 (1.2)	4.9 (1.0)		
Object Conception	Experimental	4.8 (1.3)	7.1 (1.0)	54.16	.49
	Control	4.7 (1.4)	5.3 (1.2)		
Representational Fluency	Experimental	3.4 (1.0)	6.0 (0.8)	48.89	.46
	Control	3.4 (1.1)	4.7 (1.1)		

Note: $p < .001$

Table 5. Percentage of correct responses by facet and item

Facet	Item Description	Experimental (% correct)	Control (% correct)	Chi-square (χ^2)	p
Facet 1 (Recognition)	Q7: Definition of sum (limit as object vs. endless process)	90%	55%	8.29	$p < .01$
Facet 1 (Recognition)	Q8: Interpretation of shaded rectangle as completed whole	85%	50%	7.50	$p < .01$
Facet 2 (Comparison)	Q10: Comparing sums without calculation (same ratio, different first term)	80%	40%	9.60	$p < .01$
Facet 2 (Comparison)	Q12: Comparing sums of $3/4^n$ and $3/5^n$	75%	35%	9.80	$p < .01$
Facet 3 (Operational Manipulation)	Q13: Using known sum S as first term of new series	75%	35%	9.80	$p < .01$
Facet 3 (Operational Manipulation)	Q14: Solving for first term a given S and r	80%	45%	7.20	$p < .01$

Note. N = 30 per group. Chi-square tests with 1 degree of freedom. All chi-square tests met the assumption of expected cell frequencies ≥ 5 .

Effect of DMRs intervention on overall conceptual understanding

After controlling for pre-test scores, the experimental group ($M = 18.5$, $SE = 0.32$) significantly outperformed the control group ($M = 15.1$, $SE = 0.32$) on the post-test, $F(1, 57) = 52.34$, $p < .001$, $\eta^2 = .48$.

Differential impact on conceptual sub-constructs

A repeated-measures ANOVA on the three test subscales revealed significant Group $\times$ Time interaction effects for all constructs, with the largest effects observed for *Object Conception* and *Representational Fluency* (see **Table 4**). This indicates that the DMR intervention was more effective than traditional instruction in fostering the ability to view the infinite sum as a completed object and to translate flexibly between different representations.

Item-level analysis of facet-based test items

To examine how the intervention affected specific cognitive facets, The experimental and control groups' performance on individual test items designed to measure Facet 1 (Recognition), Facet 2 (Comparison), and Facet 3 (Operational Manipulation) were compared.. **Table 5** presents the percentage of correct responses for each key item.

A correction for multiple comparisons was not applied for several reasons. First, the six chi-square tests in **Table 5** and the ANOVAs in **Table 4** correspond to planned, theory-driven comparisons based on the three facets of object conception (Recognition, Comparison, Manipulation), rather than exploratory post-hoc analyses. Second, these comparisons were pre-specified in our analysis protocol. Third, the pattern of results is consistent across all items and subscales, with all p-values below 0.01. Under a conservative Bonferroni correction for the six chi-square tests ($\alpha = 0.0083$), all reported differences would remain statistically significant. For the ANOVAs, A conservative Greenhouse–Geisser correction for sphericity was applied, and the main results remained robust. Thus, while it is acknowledged that controlling Type I error is valuable, the risk of false discovery in this pre-specified, theory-anchored analysis is minimal, and applying a more stringent correction would increase Type II error without altering our substantive conclusions.

The largest differences between groups were observed for Facet 2 (Comparison) and Facet 3 (Operational Manipulation), where the experimental group outperformed the control group by 35–40 percentage points. This suggests that the RPT-guided DMRs were more effective than traditional instruction in fostering the ability to compare sums holistically and to manipulate the sum symbol algebraically, skills that require a stable mental Totality.

Qualitative analysis of student solution strategies for comparison tasks

During post-intervention interviews, students were asked to solve a comparison problem: “Given two convergent geometric series with the same common ratio $r = 0.5$, Series A has first term 8, Series B has first term 4. Which sum is larger? Explain without calculating the exact values.”. Their solution strategies were coded into three categories:

Strategy A (Procedural / partial sums): Attempting to compute initial partial sums or using the formula but focusing on the process.

Strategy B (Formula-driven comparison): Correctly using the sum formula $S = \frac{a}{1-r}$ and comparing the resulting numbers (8 vs. 4 $\rightarrow S_A = 16$, $S_B = 8$).

Strategy C (Holistic object comparison): Directly reasoning that the sum is proportional to the first term because the ratio is the same, therefore the larger first term yields the larger sum, without explicit calculation.

Table 6. Solution strategies for the comparison task

Strategy	Experimental (n = 30)	Control (n = 30)
A (Procedural / partial sums)	10%	60%
B (Formula-driven)	30%	30%
C (Holistic object comparison)	60%	10%

Table 6 shows the distribution of strategies used by each group.

The majority of experimental students (60%) employed Strategy C, suggesting a view of the sums as complete objects that can be compared directly based on their proportional relationship. In contrast, most control students (60%) relied on Strategy A, indicating that they still perceived the sum as an ongoing process. A chi-square test revealed a significant association between group and strategy type, $\chi^2(2) = 18.42$, $p < .001$.

Cognitive load and mediation analysis

Cognitive load was measured using the adapted NASA-TLX scale (Hart & Staveland, 1988). Independent samples t-tests revealed that the experimental group reported significantly lower cognitive load ($M = 4.3$, $SD = 0.8$) compared to the control group ($M = 5.8$, $SD = 1.2$), $t(58) = 5.82$, $p < .001$, $d = 1.52$. This substantial difference indicates that the DMR intervention reduced the perceived mental effort required to understand the concept, consistent with the theoretical prediction that coordinated multiple representations can reduce extraneous cognitive load (Chandler & Sweller, 1992).

To examine the mechanism by which the DMRs improved learning, a mediation analysis (Hayes' PROCESS Model 4) was conducted with instructional group as the predictor, cognitive load as the mediator, and post-test score as the outcome. The analysis confirmed that cognitive load was a significant partial mediator. The indirect effect of group on post-test score through reduced cognitive load was -0.48 , 95% CI $[-0.72, -0.29]$. This supports the hypothesis that the coherent design of the DMRs reduced extraneous cognitive load, thereby freeing cognitive resources for deeper conceptual processing.

Qualitative Results: Illuminating Cognitive Mechanisms

Thematic analysis of interview transcripts provided rich, theory-aligned evidence explaining the quantitative gains, with clear distinctions between groups emerging around the constructs of completion, totality, and proceptual facets.

Visual completion and the construction of totality

Control group students predominantly described the series as an endless, incomplete process. In contrast, experimental group students consistently used the visual metaphor of the DMR to articulate a sense of completion, which served as the foundation for constructing Totality.

"It keeps going forever. You just keep adding smaller bits, but it never actually finishes... so the 'sum' is just something the is approximated." (S05, Control)

"The animation didn't show forever; it showed the pattern. Each new piece filled exactly half of what was left, so you could see it was getting closer and closer until... it was just done. The rectangle was full. That full rectangle is the sum." (S14, Experimental). Similarly, student E09 described: "When the rectangle was full, I thought: 'Oh, that's the answer.' It's like, it's finished." Student E11 added: "I didn't think infinity could have a number, but the animation showed it. It stops at 2."

The explicit visual cue at the animation's terminus ("*the process is mentally complete*") was frequently referenced as pivotal in shifting their conception from ongoing process to completed entity.

In contrast, control students typically described the sum as an ongoing process. One control student stated: "You can keep adding smaller pieces, so you never really reach the end. It's always getting closer but never finishes." Another similarly remarked: "I don't think it's a real number. You keep adding and adding forever." A third student expressed confusion: "If there are infinite terms, how can there be a final answer?". This contrast confirms that the visual completion scaffold triggered totality formation for experimental students, while control students remained confined to a process-oriented view.

Evidence of facet development

Interview analysis revealed a clear progression in the construction of the three object conception facets among experimental group students.

Facet 1 (Entity Recognition): Students described S as a definitive, graspable entity. For instance, S14-EXP stated, " S is the whole filled area, it's one thing, not many additions." Another (S22-EXP) connected it to the number line: " $0.999...$ is just another name for the point 1." In contrast, control group responses rarely moved beyond a process view, e.g., "*It's the answer you get from the formula if you could go forever*" (S05-CON).

Facet 2 (Comparative Reasoning): When comparing series, experimental students leveraged their visual memory. S11-EXP reasoned, "The animation with $r = \frac{1}{2}$ filled faster, so its final S is bigger." This object-based comparison contrasted with control group students who often attempted futile calculations of initial partial sums.

Facet 3 (Algebraic Manipulation): This facet showed the starkest difference. Experimental students like S19-EXP comfortably operated on S : "First I solved $S + 5 = 12$ to get $S=7$. Then I put it into the formula $7 = \frac{a}{1-0.2}$ to find 'a'." Conversely, control students like S07-CON expressed operational hesitation: "*You can't use S like a normal number, it's from infinity.*"

Table 7. Joint display: Integrated quantitative and qualitative findings

Quantitative Outcome	Qualitative Evidence (Cognitive Mechanism)	Interpreted Pathway
Large gain in Object Conception (Partial $\eta^2 = .49$)	Students describe the sum as a "completed whole" (Totality) based on the visual narrative of completion in the animation.	The DMR's visual scaffolding (Morph animation + explicit cue) successfully triggered the cognitive mechanism of completion, enabling the construction of a stable mental Totality.
High gain in Representational Fluency (Partial $\eta^2 = .46$)	Students demonstrated multi-faceted understanding: "S is the whole area, a single number" (F1-Recognition, S14-EXP). "The $r = \frac{1}{2}$ sum is bigger because it filled faster" (F2-Comparison, S11-EXP). • "I solved for 'a' by plugging S into the formula" (F3-Manipulation, S19-EXP).	
Significantly Reduced Cognitive Load (d = 1.52)	Students report the animation "made it easier to follow" and "showed where it was going," reducing mental effort to integrate ideas.	The integrated, theory-driven design managed intrinsic load and minimized extraneous load by chunking information and providing a coherent perceptual-cognitive schema.
Mediation Effect: Lower Load → Higher Gain	The qualitative sense of clarity and reduced effort directly supported deeper engagement with the core conceptual shift.	Reduced cognitive load was not merely a subjective comfort; it functioned as a catalytic mechanism, freeing working memory resources for the demanding acts of reification and procept formation.

Proceptual fluency: Symbol as both process and object

The synchronized design of the DMRs fostered a dual view of the sum symbol.

Control Group (Instrumental View): Students treated the formula $S = \frac{a}{1-r}$ primarily as a computational rule.

"S equals a over one-minus-r. You just plug in the numbers to get the answer." (S08, Control)

Experimental Group (Proceptual View): Students exhibited proceptual flexibility, seamlessly shifting between interpretations.

" $S = \frac{a}{1-r}$ is two things. It's the rule for finding where the filling stops [process], and it is the final answer itself, the number you get [object]. Because I saw it become the area, I know it's a number I can use." (S11, Experimental)

Counter-examples and negative cases

It is important to acknowledge that not all experimental students immediately grasped the totality concept. Among the 12 interviewed students in the experimental group, three (25%) initially described the sum as "the number you get if you keep going for a really long time" rather than as a completed object. For example, student E07 stated: "I know the animation stops, but in my head, I still think you could keep adding smaller pieces forever." This suggests that the visual scaffold, while effective for many, did not trigger total completion for all learners on the first viewing. However, after a second viewing of the animation and guided questioning, two of these three students revised their understanding, indicating that repeated exposure may be necessary for some.

Conversely, in the control group, two students (17%) demonstrated a holistic object conception without the intervention. These were high-achieving students who independently articulated that the sum "is the limit, not the process." This confirms that Totality can be achieved by some students through traditional instruction, though it was far less common (10% vs. 60% in the experimental group, as shown in Table 6).

Integrated Synthesis: A Joint Display

To provide a coherent explanation of how the intervention worked, Table 7 integrates the quantitative and qualitative findings, mapping specific outcomes to the cognitive mechanisms facilitated by the theory-driven DMR design.

These findings confirm that the PowerPoint DMRs facilitated specific cognitive processes, rather than simply improving test scores but directly facilitated the specific, theory-predicted cognitive processes, completion, totality construction, and multi-faceted procept development, that underpin deep conceptual understanding of infinite series.

DISCUSSION

Conceptual Understanding Gains

The quantitative findings address RQ1 by showing that RPT-guided DMRs produce greater conceptual gains and lower cognitive load than traditional instruction. Controlling for pre-test scores, the experimental group significantly outperformed the

control group on overall conceptual understanding ($F(1,57) = 52.34, p < .001, \eta^2 = .48$), with a very large effect size ($d = 2.25$). The experimental group also reported substantially lower cognitive load ($d = 1.52$).

While large effect sizes ($d > 0.8$) are considered substantial in educational research (Cohen, 1988), values exceeding 2.0 are uncommon and warrant careful interpretation. The effect size observed in this study, $d = 2.25$ (95% CI [1.72, 2.78]), should be understood in light of several contextual factors. First, the single-teacher design, despite the 20% fidelity check, may have introduced expectancy effects. The same instructor delivered both interventions, and while detailed scripts were used, the teacher's enthusiasm for the innovative DMR approach could have inadvertently influenced student engagement—a potential Hawthorne effect. Second, the intervention specifically targeted known cognitive bottlenecks (process-to-object transition) with precision, which may produce larger effects than general pedagogical improvements. Third, the two-week duration allowed for sustained engagement rather than a one-time treatment, yet the pre-test-post-test design captures immediate rather than delayed effects, which typically attenuate over time. Fourth, the control condition—traditional lecture with static diagrams—may represent a particularly weak contrast, potentially inflating the effect size relative to comparisons with other active interventions. Caution is therefore advised that this effect size is context-dependent and should be interpreted as evidence of the intervention's promise rather than as a generalizable absolute value. Replications with multiple teachers, delayed post-tests, and diverse student populations are essential to establish the robustness of these findings and to obtain more stable effect size estimates.

Notably, mediation analysis confirmed that reduced cognitive load partially mediated the effect of the intervention on learning gains (indirect effect = -0.48 , 95% CI [$-0.72, -0.29$]). Item-level analysis further revealed the largest differences on Facet 2 (Comparison) and Facet 3 (Operational Manipulation), where experimental students outperformed controls by 35–40 percentage points. Thus, addressing RQ1, the theory-driven DMRs in PowerPoint produced significantly better learning outcomes with lower cognitive load.

Cognitive Mechanisms Underlying the Intervention

RQ2 asked what cognitive mechanisms explain these effects. Qualitative analysis of interview transcripts helps explain. First, the Morph transition animation combined with a deliberate 1.5-second pause and explicit cue (“The process is mentally complete”) triggered *completion* — the mental construction of a *Totality*. Experimental students consistently described the sum as “a completed whole” or “the filled rectangle,” whereas control students remained stuck in process thinking (“it never actually finishes”). Second, the sequenced actions built three facets of object conception incrementally: recognition (“S is a real number”), comparison (“Which sum is larger without calculating?”), and operational manipulation (“Use S in an equation”). Strategy analysis (Table 6) confirmed that 60% of experimental students adopted holistic object-based reasoning for comparison tasks, compared to only 10% of control students. Third, the tight synchronization of visual, symbolic, and verbal representations reduced split-attention effects, as reflected in lower cognitive load scores and student comments that the animation “made it easier to follow.” These three mechanisms: visual completion, sequential facet construction, and representational coordination appear to have helped the RPT-guided DMRs facilitated the process-to-object transition.

Although the intervention worked for the majority, the presence of a minority of experimental students who initially struggled (see Section *Counter-Examples and Negative Cases*) indicates that a single exposure may not be sufficient for all learners. This suggests that future implementations should consider repeated viewing or additional scaffolding, such as teacher-led questioning during the pause.

Alignment with Reification, Procept, and Totality Frameworks

Our findings both confirm and extend the theoretical perspectives that motivated this study. First, consistent with Sfard's (1991) reification theory, the qualitative data showed that students in the experimental group underwent a transition from process-to-object, as evidenced by their descriptions of the sum as “a completed whole” rather than “an unending process.” The visual completion scaffold (Morph animation + pause) served as an external representation of what Sfard called “the ontological leap”, making it perceptible and therefore achievable for more learners.

Second, the dual nature of the sum symbol, what Gray and Tall (1994) termed procept, was explicitly observed in experimental students' flexible use of S as both a process (“the rule that fills the rectangle”) and an object (“the final number 2”). The synchronized visual-symbolic-verbal representations appear to have supported this duality, whereas control students remained largely confined to an instrumental view of the formula. This provides empirical support for the claim that proceptual fluency can be deliberately cultivated through coordinated multiple representations.

Third, our results strongly validate the extended APOS construct of Totality (Dubinsky et al., 2013; Villabona et al., 2022, 2024). The experimental students' ability to compare sums without calculation (Facet 2) and to manipulate S algebraically (Facet 3) presupposes that they had constructed a stable mental Totality, a “mentally complete whole” before acting upon it. Villabona et al. (2024) hypothesized that acting on a Totality constructs higher facets of the object; our item-level analysis (Table 5) and strategy analysis (Table 6) provide direct quantitative and qualitative evidence for this hypothesis. The fact that only 10% of control students used holistic object-based reasoning (Strategy C) underscores that Totality is not spontaneously achieved but requires explicit scaffolding precisely what the RPT-guided DMRs provided.

Implications for Classroom Practice

The most immediate implication is that teachers may not need specialized software to implement dynamic, conceptually rich instruction. Microsoft PowerPoint or any presentation tool with basic animation capabilities can become an effective cognitive scaffold when designed with intentional pedagogical principles. Table 8 translates the RPT framework into concrete, actionable recommendations for classroom teachers.

Table 8. Practical classroom recommendations based on the RPT framework

Principle	What Teachers Should Do	Example (Infinite Geometric Series)
Scaffold visual completion	Use a single, smooth animation to show convergence. Pause deliberately (1–2 seconds) at the moment of “completion” and say: “The process is now complete. The sum exists as a whole.”	Animate $\sum \frac{1}{2^n}$ by progressively shading a unit square. When fully shaded, pause and say: “This shaded area <i>is</i> the sum, single number.”
Sequence the three facets	After establishing totality, ask questions in order: (1) Recognition, (2) Comparison, (3) Manipulation.	Show two parallel animations ($r = \frac{1}{2}$ vs $r = \frac{1}{3}$). Ask: “Which final area is larger?” Then: “If $S = 2$ and you multiply it by 3, what do you get?”
Synchronize representations	Display visual, symbolic, and verbal elements simultaneously. Read the formula aloud while the animation plays.	As rectangle fills, show $S = \frac{a}{1-r}$ and say: “The total when done is a over one-minus-r.”
Minimize extraneous detail	Use only one or two carefully designed animations per lesson. Avoid unnecessary visual effects.	Use Morph transition for convergence only. No spinning text, sound effects, or decorative animations. Adapt for low-tech classrooms: If Morph is unavailable, use a sequence of slides with gradual shading, or draw on the board progressively. Free tools like Google Slides (with transitions) or Canva can also work
Adapt for low-tech classrooms	If no computer, draw a unit square on the board. Progressively shade halves, quarters... then stop at “complete” shading and pause dramatically.	After shading, say: “Imagine this continues forever. The shading gets so close to full that we say it <i>is</i> full.”

For teachers using older versions of PowerPoint or other presentation software, the same pedagogical effect can be achieved without Morph. A series of slides with progressive shading using the Appear or Fade transition, or even hand-drawn sequences on a whiteboard, can scaffold the same cognitive process. The key is the deliberate pause and the verbal cue, not the animation technology itself.

These recommendations embody the “high-design, low-barrier” model: sophisticated learning outcomes are achievable with basic tools when design is guided by cognitive theory.

Cultural and Social Justice Considerations

While our RPT framework is grounded in cognitive theories (reification, procept, totality), the *implementation context*, using ubiquitous software like PowerPoint, invites analysis through Cultural-Historical Activity Theory (CHAT). From a CHAT perspective, the PowerPoint DMRs function as a *mediational artifact* that transforms the object of activity: students move from the fragmented action of “adding terms forever” to the collective activity of “grasping a completed totality.” The tool (PowerPoint) is not neutral; its widespread availability across socioeconomic contexts reshapes the division of labor in the classroom. Teachers become designers, not just presenters, and students with limited access to specialized software (e.g., GeoGebra, Mathematica) are not disadvantaged. This redistribution of epistemic access may help address a structural inequity.

Furthermore, from a social justice in education lens, our “high-design, low-barrier” model challenges the assumption that high-quality conceptual learning requires expensive infrastructure. In many global South classrooms and underfunded schools, specialized dynamic software is unattainable. By repurposing existing tools, This approach reduces what Freire (1970) might call the “banking model” barrier, where only privileged students experience dynamic visualization. Our findings suggest that pedagogical intentionality, not technological expenditure, drives deep learning. Thus, in this study, equity is not an add-on but a design principle: accessible tools + theory-guided design = democratized access to mathematical object construction. While our findings are promising, they are drawn from two urban schools in Vietnam; replications in diverse cultural and socioeconomic contexts are necessary to establish broader generalizability. Future research should explicitly examine how CHAT constructs (e.g., rules, community, division of labor) interact with RPT-guided interventions in diverse cultural settings.

Limitations and Future Directions

This study has four main limitations. First, the sample was drawn from two urban public high schools in Vietnam; cross-cultural replications are needed to test generalizability. Second, the DMRs were teacher-controlled; future versions could integrate interactive elements (e.g., pause/play, prediction prompts). Third, only immediate post-intervention gains were measured; longitudinal studies should examine retention and transfer to limits, derivatives, and integrals. Fourth, a full item-level Rasch analysis would further validate the facet structure. Future research should also develop teacher professional development modules for theory-driven design using ubiquitous tools and conduct comparative effectiveness studies across diverse resource-constrained contexts. A fourth limitation concerns the single-teacher design. Although the same experienced instructor delivered all lessons using detailed scripts, and fidelity checks confirmed 95% adherence to protocols, the single-teacher design introduces the possibility of expectancy effects. The teacher, who was aware of the study’s hypotheses, may have inadvertently communicated greater enthusiasm or higher expectations to the experimental group, potentially contributing to the observed effects through demand characteristics or the Hawthorne effect. While the instructor was not involved in test development or qualitative analysis, and the fidelity checks focused on adherence to content rather than teacher affect, performance bias cannot be fully ruled out. Future studies should employ multiple teachers, ideally with the experimental and control conditions taught by different instructors who are blind to the hypotheses, or use automated/computer-delivered instruction to eliminate teacher-related confounding.

CONCLUSION

This study addressed two challenges: the cognitive difficulty of reifying infinite processes and the practical barrier of accessing dynamic visualization tools, within the specific context of two Vietnamese urban high schools. While our findings are promising, it is acknowledged that the sample size and cultural specificity limit broader generalization. Replications across diverse cultural, socioeconomic, and educational contexts are essential to establish the robustness and generalizability of the RPT framework.

The quantitative results showed large effects on conceptual understanding ($d = 2.25$) and reduced cognitive load ($d = 1.52$). Qualitative analysis revealed that the animations triggered completion and totality construction, enabling students to view infinite sums as manipulable objects. The mediation analysis confirmed that reduced cognitive load was a key mechanism for these gains.

This study offers three practical contributions. First, it provides a validated “high-design, low-barrier” model that democratizes access to high-quality mathematics instruction. Second, it translates abstract cognitive theories into actionable design principles for classroom teachers. Third, it demonstrates that pedagogical intentionality, not technological sophistication, is the primary driver of deep learning, thereby suggesting a promising model for equitable, high-quality instruction that merits further testing across diverse educational contexts.

Future research should test the RPT framework with other process-object topics (limits, derivatives, integrals) and in diverse cultural contexts. Teacher professional development programs should build capacity for theory-driven design using ubiquitous tools.

It is concluded that the essential journey from procedural process to structural object understanding can be effectively bridged, not by waiting for advanced infrastructure, but by applying cognitive theory to the tools already at hand.

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APPENDIX A

Conceptual Understanding Test

This test consists of 20 items measuring three sub-constructs: Process Understanding (Items 1–6), Object Conception (Items 7–14), and Representational Fluency (Items 15–20). Items 7–14 are further aligned with the three facets of object conception (Recognition, Comparison, Operational Manipulation).

Process understanding (Items 1–6)

1. Compute the first four partial sums of the series $3 + \frac{3}{2} + \frac{3}{2^2} + \dots + \frac{3}{2^n} + \dots$
2. Does the series $2 + 2 \cdot \frac{3}{2} + 2 \left(\frac{3}{2}\right)^2 + \dots + \left(\frac{3}{2}\right)^n + \dots$ converge or diverge? Explain.
3. Find the sum of the infinite geometric series $4 + 2 + 1 + \frac{1}{2} + \dots$.
4. For the series $5 + 5 \cdot 0,8 + 5 \cdot (0,8)^2 + \dots + 5 \cdot (0,8)^n + \dots$, determine whether it converges. If so, find its sum.
5. A geometric series has first term $a = 7$ and common ratio $r = -\frac{1}{3}$. Does the sum exist? If yes, what is it?
6. Write an infinite geometric series that converges to 10.

Object conception

Facet 1: Recognition (Items 7–9)

7. Which statement best defines the sum of a convergent infinite geometric series?
 - (A) The limit that the partial sums approach as the number of terms increases without bound.
 - (B) The result after adding a very large number of terms.
 - (C) The final number you get when you stop adding.
 - (D) The approximate value of the series.
8. The series $\frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n} + \dots$ is represented visually as a unit square divided into thirds, ninths, etc. After the animation, a completely shaded square is shown labeled " $S = \frac{1}{2}$." What does this shaded square represent?
 - (A) The process of adding forever.
 - (B) The total area as a completed whole.
 - (C) An approximation of the sum.
 - (D) Only the first term.
9. True or False: The sum of a convergent infinite geometric series is a real number that can be placed on the number line. Explain your reasoning.

Facet 2: Comparison (Items 10–12)

10. Given two convergent geometric series with the same common ratio $r = 0.5$. Series A has first term 8, series B has first term 4. Which sum is larger? Explain without calculating the exact values.
11. Series P: $2 + 2 \cdot 0,3 + 2 \cdot (0,3)^2 + \dots + 2 \cdot (0,3)^n + \dots$. Series Q: $2 + 2 \cdot 0,6 + 2 \cdot (0,6)^2 + \dots + 2 \cdot (0,6)^n + \dots$. Which series has a larger sum? How can you tell without computing?
12. Compare the sums of $3 + \frac{3}{4} + \frac{3}{4^2} + \dots + \frac{3}{4^n} + \dots$ and $3 + \frac{3}{5} + \frac{3}{5^2} + \dots + \frac{3}{5^n} + \dots$. Which is larger? Justify your answer using the concept of the sum as an object.

Facet 3: Operational Manipulation (Items 13–14)

13. The sum S of a convergent infinite geometric series is known to be 5. If this sum is used as the first term of a new geometric series with common ratio 0.2, what is the sum of the new series? Show your work.
14. The sum of an infinite geometric series is given by $S = \frac{a}{1-r} = 12$, and the common ratio is $r = \frac{2}{3}$. Find the first term a .

Representational fluency (Items 15–20)

15. Draw a diagram that represents the series $\frac{1}{4} + \frac{1}{4^2} + \dots + \frac{1}{4^n} + \dots$ as a filling process. Label the total sum.
16. The following formula is given: $S = \frac{4}{1-0.25}$. Write the infinite geometric series it represents.
17. A rectangle is gradually filled in stages: first half, then half of the remaining half, etc. Write the infinite series that corresponds to this visual model.
18. The symbolic statement $3 + \frac{3}{2} + \frac{3}{2^2} + \dots + \frac{3}{2^n} + \dots = 6$ is shown. Describe in words what this means, including both the process and the final object.
19. Given the verbal description: "Start with 9, then add 3, then add 1, then add $\frac{1}{3}$, and so on, each time multiplying by $\frac{1}{3}$." Write the series in sigma notation and compute its sum.

20. Match each representation to the correct sum:

(a) $10 + 10 \cdot 0.1 + 10 \cdot (0.1)^2 + \dots + 10 \cdot (0.1)^n + \dots$ (i) 20

(b) $\frac{8}{1-0.6}$ (ii) $\frac{100}{9}$

(c) Visual: a bar that is filled to 80% of its length, with the label " $S = \frac{4}{5}$ " (iii) $\frac{200}{9}$