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Students' conceptualization of Pythagorean theorem (in RME): Examination with APOS

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ABSTRACT

This study investigates how eighth-grade students conceptualize the Pythagorean theorem through an instructional design grounded in realistic mathematics education (RME), analyzed using the APOS (action, process, object, schema) theoretical framework. Employing a design-based research methodology, a classroom activity was developed in which students used square tiles and right triangles to explore and ultimately discover the relationship now known as the Pythagorean theorem. The study was conducted with an entire eighth-grade classroom, and data-including group work observations and student interviews- were analyzed to trace students' transitions across APOS stages. Three focal student cases were examined in depth to illustrate diverse developmental trajectories. The results indicate that while all students began with physical manipulation and informal reasoning, two students progressed to object-level understanding. The findings provide insights into how RME-based tasks can foster conceptual development and highlight areas for refining instructional design to better scaffold students' transitions across cognitive stages.

Keywords: APOS theory, individual analysis, Pythagorean theorem, realistic mathematics education, mathematical understanding

INTRODUCTION

Concept formation is a cognitive product that emerges through abstraction and interaction with new experiences (Skemp, 1986). In mathematics education, conceptual understanding is essential for fostering reasoning and problem-solving skills (Rittle-Johnson & Schneider, 2015). It allows learners to grasp structures and relationships in mathematics, supporting meaningful and enduring learning (Baroody et al., 2007; Hiebert & Grouws, 2007). Conceptual learning goes beyond memorization by enabling students to internalize mathematical logic, connect ideas, and tackle complex problems (Star, 2005; Rittle-Johnson & Schneider, 2015). It also boosts engagement and intrinsic motivation (Boaler & Staples, 2008). Its importance is widely recognized, as emphasized in the NCTM's (2020) recommendations for mathematics education.

Despite this, students often struggle with understanding and applying mathematical concepts (Stacey, 2011; Verschaffel et al., 2019), largely due to instructional approaches that overlook conceptual development (Boaler, 2016). This neglect limits problem-solving abilities and weakens critical thinking and comprehension (Baroody et al., 2007; Prawat, 1989). This study explores how eighth-grade students develop the Pythagorean relationship using realistic mathematics education (RME) and analyzes their understanding through the APOS (action, process, object, schema) framework, highlighting the significance of concept-driven instruction.

Problem Statement

Research consistently shows that connecting mathematical concepts to real-life contexts significantly enhances students' understanding (Boaler, 2016; Verschaffel et al., 2019). Real-world applications make mathematics more meaningful and increase student motivation (Boaler, 2016; Verschaffel et al., 2019). When students see the relevance of math in daily life, they better internalize and apply concepts (Cai & Lester, 2019). Realistic contexts also promote problem-solving skills by engaging students in multi-step, complex reasoning (Gravemeijer, 1994). Empirical studies confirm that instruction embedded in real-life contexts deepens conceptual understanding and supports long-term academic success (Cai & Lester, 2019; Van den Heuvel-Panhuizen, 2020). RME, developed by Hans Freudenthal, is a widely adopted approach that emphasizes the learning of mathematics through meaningful, everyday contexts (Van den Heuvel-Panhuizen, 2020). Freudenthal (1968) viewed mathematics as a human activity rooted in real-world problem-solving and evolving through mathematization. Realistic Mathematics Education (RME) emphasizes

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both theoretical and practical understanding, thereby supporting students' conceptual learning (Freudenthal, 1968; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 2020).

Among the most foundational mathematical concepts taught in school is the Pythagorean theorem. It has long attracted interest due to its many proofs and central role in mathematics (Alsina & Nelsen, 2011; Bellos, 2012). Understanding this theorem underpins key areas such as geometry, trigonometry, and analytic geometry (Boaler, 2016) and supports conceptual knowledge of squares and square roots (Verschaffel et al., 2019). It also prepares students for spatial reasoning and real-world applications, especially in engineering and architecture (NCTM, 2020). Moreover, it often introduces students to related ideas such as geometric similarity (Clements & Battista, 1992; Van de Walle, 2007).

This study contributes to the limited body of research investigating the cognitive processes underlying the conceptualization of the Pythagorean theorem within the framework of RME. By integrating APOS theory and a design-based research (DBR) methodology, the study provides a nuanced understanding of how students transition between different levels of mathematical abstraction. While previous research has addressed conceptual development in algebra and functions using APOS, few studies have explored geometric concepts like the Pythagorean theorem through the lens of RME and in real classroom environments. The detailed case analyses illuminate the varied learning trajectories students may follow and offer insights into how instructional design can support these pathways. Moreover, the study has practical implications for mathematics teachers seeking to design rich, contextual learning experiences that scaffold students' conceptual growth. In contexts where traditional procedural teaching dominates, this study demonstrates how realistic and student-centered approaches can foster deeper mathematical reasoning and understanding.

Despite its importance, students often struggle with the conceptual foundations of the theorem (Guisasola & Morentín, 2007; Lian et al., 2023). Difficulty in grasping the geometric logic behind it can hinder learning (MIT Blossoms, 2023) and negatively affect problem-solving when connections between geometric representations and underlying relationships are weak (English, 1997; Richland et al., 2012). While the literature discusses various teaching approaches, there is a lack of empirical research on how students cognitively construct the Pythagorean relationship within a Realistic Mathematics Education (RME) setting. This study aims to address that gap.

Studies Using Realistic Mathematics Education

A growing body of research emphasizes the value of RME in fostering deep mathematical understanding. RME situates mathematical learning in meaningful, real-life contexts, enabling students to actively construct knowledge through realistic problem situations (Gravemeijer, 1994; Van den Heuvel-Panhuizen, 2020). Studies show that RME-based instruction enhances students' engagement, motivation, and reasoning abilities by encouraging them to mathematize everyday experiences (Boaler, 2016). Empirical research further demonstrates that RME contributes to students' conceptual development in topics such as proportional reasoning, geometry, and spatial visualization (Cai & Lester, 2019; Van de Walle, 2007). Through guided reinvention and contextual modeling, RME supports the transfer of mathematical knowledge beyond the classroom (Verschaffel et al., 2019).

Studies Using the APOS Framework

APOS theory, developed by Dubinsky et al. (2013), has been widely used to explore the stages of students' conceptual development in mathematics. Research reveals that students often struggle to transition from process to object-level understanding, especially when dealing with abstract mathematical concepts such as functions, limits, and geometric relationships (Cottrill et al., 1996; Dubinsky & Wilson, 2013; Weller et al., 2009). Recent studies have proposed an intermediate "coherence" stage to better capture students' cognitive shifts between stages (Borji et al., 2023a; Sfard, 2019; Tall et al., 2021). While APOS theory has been applied to algebra, calculus, and trigonometry (Setyaningsih & Safi'i, 2022), few studies have examined its integration within an RME-based learning environment. This study seeks to address this gap by combining APOS analysis with realistic modeling tasks focused on the Pythagorean theorem.

Research Purpose and Questions

The present study investigates how eighth-grade students construct the concept of the Pythagorean relationship within a RME environment, employing the APOS framework to analyze their cognitive learning processes. This inquiry was designed and implemented through a DBR approach, which allowed iterative development and refinement of context-rich instructional tasks grounded in RME principles. The Pythagorean theorem occupies a prominent place in mathematics curricula worldwide, and this study aims to uncover how students' conceptualizations evolve through meaningful engagement in such design-informed learning experiences. The study is expected to contribute to both theoretical and practical domains by shedding light on the interplay between RME, design-based implementation, and cognitive development as theorized by APOS.

Research questions

1. How do eighth-grade students construct the concept of the Pythagorean relationship within a RME setting?
    a. At what levels and in what ways do students' conceptualizations manifest according to the APOS theoretical framework?
2. How does the design-based RME instructional environment support or constrain students' progression through APOS stages?

Table 1. The participant profile

Group	Students	Selected participant	Gender	Achievement level
1	S1, S6, S10, & S11	S1	Female	High
2	S2, S5, S7, & S12	S2	Male	Very high
3	S3, S8, S9, & S13	S3	Male	Medium

METHOD

Research Design

This study employed a DBR methodology to iteratively develop, implement, and analyze a learning environment aimed at fostering students' conceptual understanding of the Pythagorean theorem. DBR involves systematic cycles of designing instructional activities, implementing them in authentic classroom settings, and analyzing the outcomes to refine both theory and practice (Gravemeijer & Cobb, 2006, 2013; Plomp & Nieveen, 2013).

The instructional design was informed by the principles of RME, embedding mathematical learning within meaningful, context-based tasks. Students' cognitive constructions and conceptual development were examined through the APOS theoretical framework, allowing the researchers to analyze learning as a developmental process. Although this study reflects the initial design-implementation-evaluation cycle, it integrates elements of iterative refinement through expert consultation and pilot testing prior to the main classroom intervention. The purpose of the DBR approach here was not only to evaluate student learning, but also to generate theoretical insights into how design principles grounded in RME can support transitions across APOS stages.

Participants

The study was implemented in a classroom of 16 students at a mid-level public school, with students grouped into four heterogeneous groups of four. Johnson and Johnson (1989) emphasized that heterogeneous groups yield better performance and enhance peer learning opportunities. Accordingly, group formation considered various factors, including students' report card grades, readiness assessment scores, results from the most recent mock exam, teachers' opinions, peer interaction levels, and observed mathematical skills. To gather rich and detailed data on students' conceptual understanding of the Pythagorean relationship, maximum variation sampling–a purposeful sampling method- was employed (Patton, 1987). One student from each group was selected, representing *different achievement levels and strong communication skills*. This strategy aimed to capture how the learning environment supported students across various levels. The participant profile is summarized in **Table 1**.

The students attended a school located in a rural area. In this study, data were collected from three groups, as one group was incomplete due to a member's absence during the session. The teacher-researcher had been teaching these students since the fifth grade and was well-acquainted with their learning profiles. Throughout this time, the teacher had consistently implemented instructional practices grounded in the principles of RME. Additionally, the teacher had previously conducted an academic study with the same students in an earlier grade level, indicating that participants were familiar with RME-based learning environments. Given this context, the current study was situated within a DBR framework, allowing for iterative refinement of instructional tasks based on observations of students' engagement and conceptual development.

The Role of the Researcher

Contemporary educational perspectives emphasize that teachers should not merely transmit knowledge but also act as researchers who treat the classroom as a learning laboratory (Lewis et al., 2006; Timperley et al., 2007). The teacher-researcher model allows educators to continuously observe their instructional practices, tailor pedagogical strategies based on student needs, and create more effective learning environments (Davidson, 2017; Goos, 2014). Furthermore, this model promotes professional growth not only for the teacher conducting the research but also for their colleagues (Botha et al., 2023).

Teacher-researchers are able to make evidence-based decisions by analyzing findings from their own classroom practices and are thus better equipped to refine their instruction (Timperley et al., 2007). The culture of collaborative learning among teachers, supported by knowledge sharing, enables ongoing updates to pedagogical approaches (Robutti et al., 2019). In this study, the researcher systematically evaluated classroom experiences and developed instructional practices that supported students' conceptual learning. The teacher's ability to critically reflect on classroom implementation enhanced professional development and strengthened pedagogical competencies (Goos, 2014). Additionally, interactions within the school environment and among colleagues contributed to teacher motivation and the creation of a more effective learning atmosphere (Collie et al., 2012).

Design and Implementation of the Contextual Problem

Researchers have emphasized that when students are unable to apply what they learn to real-life situations, mathematical concepts tend to lose their meaning (Freudenthal, 1968; Gravemeijer, 1994). Therefore, it is essential that school-based mathematical concepts are contextualized through real-world situations. This alignment enhances meaning-making by integrating the subject matter with real-life applications (Berns & Ericson, 2001). In the current study, the contextual problem was developed over a four-week period. Multiple problems were drafted, and the one deemed most meaningful and conceptually rich for the students was selected. After expert review and pilot testing with a class other than the research participants, necessary revisions were made. The finalized problem was implemented during the intervention. At the beginning of the lesson, the problem was introduced via a smart board, and printed problem sheets were distributed to each group. The teacher, in a fictional context, asked for the students' help in solving the problem to fulfill his grandfather's will. Materials like color-coded poster boards divided into unit squares and right triangle cutouts with marked edge lengths were also provided. The full version of the contextual problem used in the study is included in **Appendix A** and **Appendix B**.

Data Collection Process and Implementation

The instructional process was designed following the principles of RME. A natural classroom environment was created to encourage students to think freely in collaborative groups. Students' mathematization processes were guided and supported throughout the lessons. Prior to instruction, a readiness assessment was administered to evaluate students' prior knowledge related to the Pythagorean relationship. In line with the DBR methodology, the instructional design process included preliminary stages of needs analysis, expert consultation, and pilot implementation. Contextual problems served as the structured entry point for conceptual development. The central instructional task, known as the tile problem, was designed to engage students in exploring the Pythagorean relationship through contextual reasoning and physical modeling. By using square tiles to cover triangular spaces, students were expected to visualize and quantify the areas of squares on each side of a right triangle, thereby constructing a conceptual understanding of the Pythagorean theorem. The task aimed to bridge students' informal spatial reasoning with formal mathematical generalization in line with the RME and APOS frameworks.

A classroom activity was developed in which students used square tiles and right triangles to explore and ultimately discover the relationship now known as the Pythagorean theorem. Due to limitations in available resources at the school, unit cubes were used as the primary physical material for modeling square tiles. This decision was made based on their abundance and versatility, allowing students to construct and manipulate square areas with ease. The use of these concrete materials supported spatial reasoning and hands-on exploration in line with RME principles. The problem-solving process formed the primary data source of the study. After solving the problems in groups, each group presented its solution, followed by a class-wide discussion aimed at reaching a consensus. This entire process spanned two class hours.

The classroom implementation phase was carefully documented as part of the iterative DBR cycle. All group discussions were recorded using audio devices placed on each table, as well as a fixed camera capturing the general classroom setting. Transcriptions of these recordings were included in the data set. After the problem-solving session, worksheet activities were distributed to reinforce the Pythagorean concept. These worksheets were collected for analysis.

Individual interviews were conducted with the selected focal students from each group, one day after the problem-solving session. Each interview lasted approximately one class hour and was held in a comfortable setting, where students were encouraged to think aloud while solving similar problems using the same materials. These interviews provided insight into the students' cognitive processes. The aim was to uncover the reasoning behind their answers and understand how they constructed their mathematical thinking. The researcher also took observational field notes throughout the process, which were included in the data set. Additionally, homework assignments were administered to further assess students' understanding outside the classroom. The primary data collection tools aligned with DBR triangulation principles included:

(1) classroom observations and field notes,

(2) audio and video recordings (transcripts),

(3) group and individual worksheets, and

(4) individual interviews.

Data Analysis

The cognitive processes involved in concept formation are critical for enabling students to develop deep understanding and apply their knowledge across diverse contexts (Baroody et al., 2007). Several theoretical frameworks have been proposed to explain such processes (Boaler, 2016; Rittle-Johnson & Schneider, 2015; Skemp, 1986), among which the APOS theory has received significant attention.

In this study, the data analysis process was closely aligned with the DBR methodology. The goal was not only to examine students' conceptual development but also to inform and refine instructional design. Data derived from audio recordings, interview transcripts, group worksheets, and student work during individual interviews were analyzed using content analysis.

The transcribed data were first coded based on the APOS theoretical framework. After initial coding, emerging codes were examined for thematic relationships, which were then categorized accordingly. APOS theory functioned as the central analytical framework throughout the analysis process, while the DBR perspective allowed these findings to be fed back into the instructional design for potential future iterations.

Prior to the instructional intervention, an initial genetic decomposition of the Pythagorean theorem was conducted (see **Figure 1**). Following expert consultation, this decomposition was refined and finalized. The genetic decomposition not only informed the design of instructional activities but also helped anticipate how students might conceptually build the Pythagorean relationship during the learning process. In this way, the APOS framework served as both a predictive and analytical tool, consistent with the dual aims of DBR.

Based on this decomposition, expected behaviors for each APOS stage were identified as follows:

1. **Action:** Students construct squares on the sides of a right triangle and fill them with equal-sized units. They calculate the number of units needed to fill each square and internalize this experience through repetition across different triangles.
2. **Process:** At this stage, students begin to coordinate the concepts of squaring, square root, and area. They compute the area of the squares and observe that the sum of the areas on the legs equals the area on the hypotenuse.

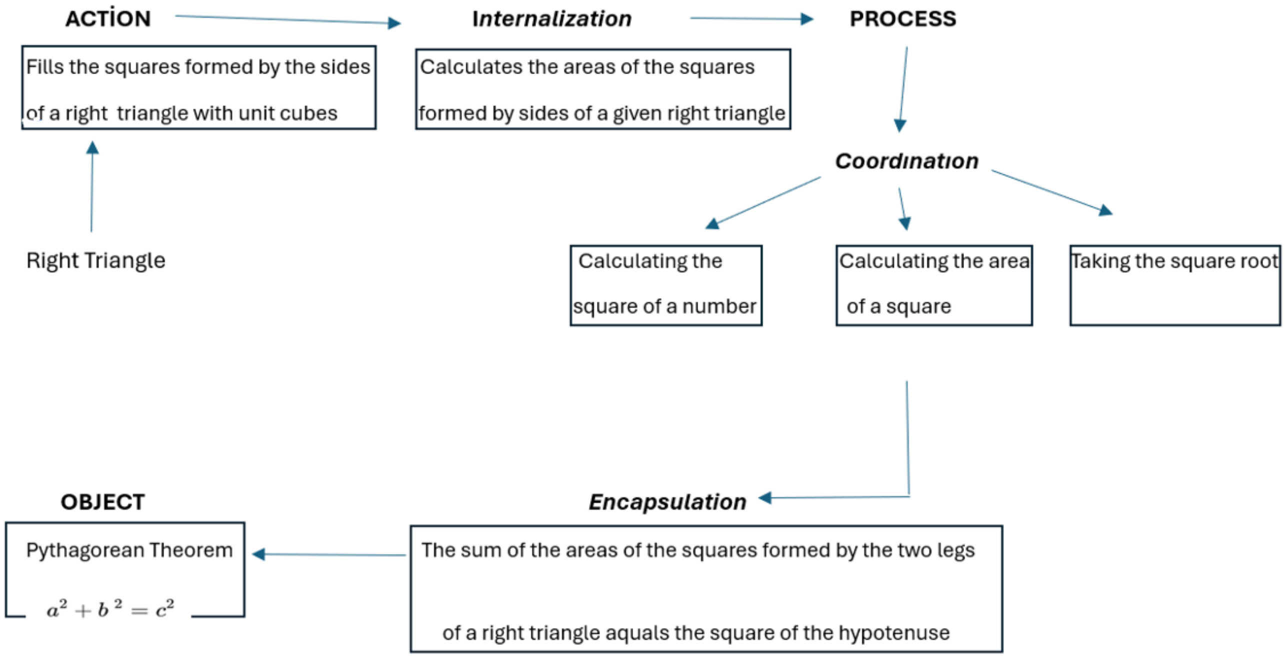


Figure 1. Genetic differentiation of the Pythagorean theorem concept (Source: Authors' own elaboration)

3. **Object:** Students encapsulate the relationship "the sum of the areas of the squares on the legs equals the area of the square on the hypotenuse" and apply this generalization to solve problems. This indicates that they have constructed an object-level understanding of the Pythagorean theorem.

Validity and Reliability

To ensure the trustworthiness of the study, various strategies were employed to strengthen both internal and external validity. Expert review was used to increase internal validity, particularly during the development of data collection instruments and the analysis process (Creswell, 2013; Lincoln & Guba, 1985). Three content experts reviewed the entire process, from design to data interpretation. Methodological triangulation was also used–through interviews, classroom observations, audio recordings, student worksheets, and field notes–to enhance credibility.

To ensure transferability, direct quotations from students were included without interpretation, and rich, thick descriptions were provided. For dependability, the contextual problem was carefully designed based on the principles of RME, supported by an in-depth literature review. During the development phase, feedback from two mathematics education experts was incorporated. After pilot implementation and necessary revisions, the final version of the problem was administered in the main study.

The initial genetic decomposition was also prepared using the APOS framework and expert input. This served as a foundation for designing the instructional process. Post-implementation, this decomposition was finalized based on classroom observations and student responses. Confirmability was addressed by reanalyzing the data at different times by the primary researcher and conducting an independent coding process by another mathematics education expert. The inter-rater agreement was calculated at 78%, which, according to Yıldırım and Şimşek (2016), is within acceptable limits (above 70%). Therefore, the validity and reliability of the study were rigorously supported through expert consultation, triangulation, transparent data collection, and cross-verification of coding procedures.

FINDINGS

The findings are organized according to the APOS theoretical framework and structured to address the research questions regarding students' construction of the Pythagorean relationship. Differences observed in the conceptual construction process constitute the core of the study. Findings were synthesized from both group work and individual interviews with the participants. In order to provide a comprehensive understanding, detailed findings are presented for student S1 and student S2, while data from student S3 are summarized more concisely. Findings on the construction process of the Pythagorean relationship will be presented.

Student 1 (S1)

To provide a comprehensive view of S1's conceptual development, the analysis is divided into group work and individual interview stages, aligned with the APOS framework.

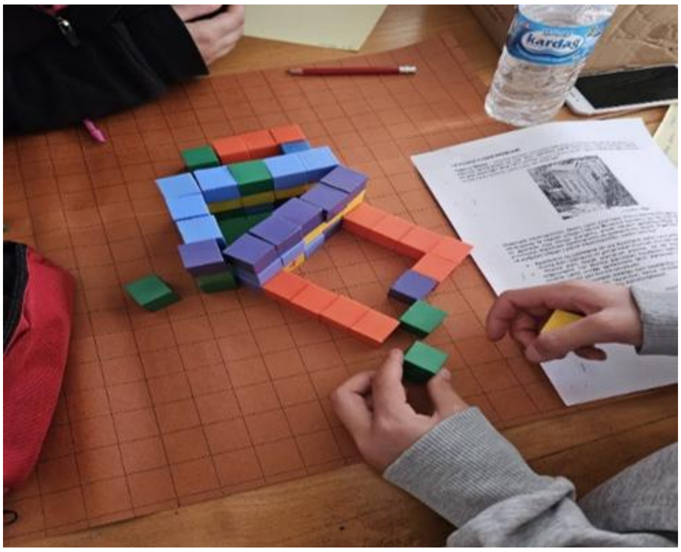


Figure 2. The physical model constructed by S1 and group members, showing squares attached to the sides of a right triangle with side lengths of 3, 4, and 5 units (Source: Field study)

Group work–Action phase (S1)

At the start of the task, S1 and peers misinterpreted the problem as constructing a building interior, attempting to build a triangular prism using units. This reflected their position in the action phase, relying on physical manipulation and intuition without a clear grasp of the mathematical objective.

S1: I'll just eyeball it … Let's make a triangular prism.

S11: How do we make a triangle?

T: Try modeling it with these units–Think cafeteria spaces outside the triangle.

This teacher prompt helped shift their focus. Initially, S1 envisioned living areas inside the triangle, but upon realizing the squares were to be built externally, the group reframed their understanding:

S1: Haaa ... We were trying to make a sitting room inside. Okay, we'll do it outside. Let's do it properly ... We won't have enough ... Lets borrow some material from others? It's probably related to a future project ... It will be square? How are we going to do this?

They began modeling square areas on the triangle's sides using units, working with triangle cutouts labeled 3, 4, and 5 units (see **Figure 2**). This adjustment marked a cognitive transition from unstructured manipulation toward more purposeful modeling. This phase exemplifies the early, often messy nature of RME-based problem-solving, where cognitive conflict serves as a catalyst for deeper engagement and movement toward the process phase within the APOS framework.

Overall, S1 and peers struggled to comprehend the task at first. Their misinterpretation-constructing interior living spaces instead of external square areas- was a key source of confusion. Their first action was to build a prism-shaped structure, but with the researcher's support, they began to focus on the key mathematical aspect: building squares on the sides of the triangle. This transition demonstrates movement from action to process within the APOS framework. Importantly, this initial struggle is consistent with the nature of RME-based tasks for problem based learning, which are intentionally designed to provoke cognitive conflict and require meaningful student engagement before resolution.

At the start of the task, S1 and peers misinterpreted the problem as constructing a building interior, attempting to build a triangular prism using units. This reflected their position in the action phase, relying on physical manipulation and intuition without a clear grasp of the mathematical objective. This initial misconception aligns with a design goal of the modeling task-provoking cognitive conflict. The ambiguity in the real-world context was purposefully integrated to stimulate group discussion and negotiation, a key component of RME. The teacher's intervention-framing the squares as exterior seating areas–redirected the students' thinking. This moment marked a successful transition initiated by both the contextual design of the task and the structured visual supports.

Group work–Process phase (S1)

As the activity progressed, S1 and peers began to demonstrate conceptual development, transitioning from action to process-level thinking within the APOS framework. They constructed squares on each side of a 3-4-5 triangle using units–5 along the hypotenuse, 4 along one leg, and 3 along the other–indicating that they had internalized the task structure. However, the group mistakenly placed an extra unit at the shared vertex of two triangle legs, leading to a miscalculation in the total tile count. The following dialogue illustrates their reasoning:

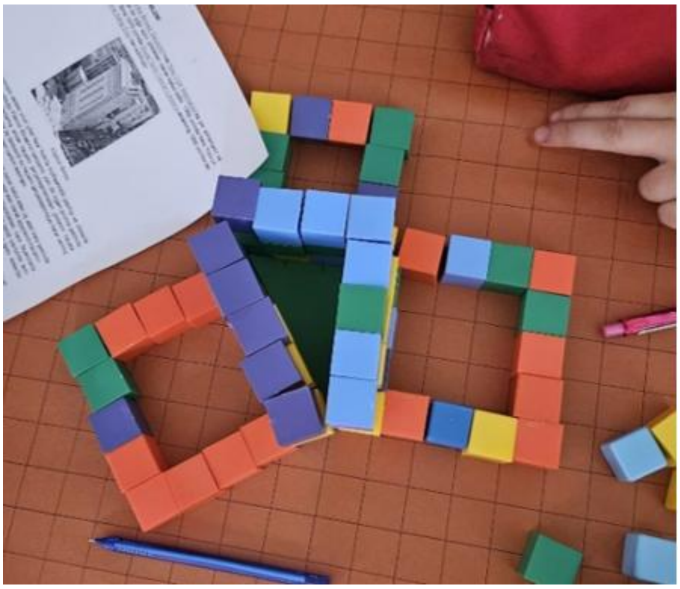


Figure 3. Erroneous model created by S1's group, showing the extra material placed at the vertex of the triangle (a visual indicator of conceptual refinement during the process phase) (Source: Field study)

S1: Teacher, when it says 'surround it with trees', which part are we decorating? The whole board?

T: It's not about trees. These are green areas. You're supposed to calculate tile coverage.

S1: Okay then ... multiples of 4 and 5 ... something divisible by both ... maybe 1?

T: The question says you'll tile this area. How would you figure out how many tiles to order?

S1 began computing tile counts by squaring side lengths:

S1: This one's 25, this one's also 25 … and this one's 16. What's the greatest common divisor of 25 and 16? ... Not 2 …

T: Why are you looking for a common divisor?

S1: So that they all divide evenly ...

This interaction reflected an evolving understanding, with S1 shifting from additive strategies to squared area reasoning. Still unsure of unit size and meaning, he correctly associated the side lengths (3, 4, 5) with their areas (9, 16, 25). Later, when calculating again:

T: If you fill this square, how many tiles would you use?

S1: 25.

T: And here?

S1: 16.

T: And here?

S1: 25. Didn't we make them equal?

S11: Yeah, we made them equal.

This dialogue shows S1 beginning to view the squares as distinct, measurable areas (**Figure 3**). Still, the earlier placement of an extra unit at the triangle's vertex caused overcounting. The teacher intervened to clarify:

T: You placed an extra square on the corner–it doesn't count; it's already occupied. How many tick marks are here?

S1: 3.

T: And here?

S1: 4 … Oh right, there are four.

T: Let's remove the extra pieces.

S1: Then this one and that one should go. Yes, now place just one here. We had 5 … so 25, 16, and 9–that's 50 tiles.

S1's consistent use of squaring operations and ability to revise his errors through dialogue and visual feedback indicate that he had fully entered the *process phase (APOS-2*). Additionally, his correction of S11's additive reasoning-"area is not adding sides, it's side times side"–revealed increasing conceptual clarity. These insights, scaffolded by materials and teacher interaction, mark a growing readiness for transition into the *object phase*, where the Pythagorean relationship is treated as an abstract mathematical entity.

At this point, S1 clearly demonstrated coordination between the *mathematical operation of squaring* and the *geometric representation* of area. His ability to calculate and relate the square areas on each side of the triangle indicates a strong alignment with the *process stage (APOS-2),* where learners begin to internalize mathematical operations as coherent mental actions. Importantly, S1 identified the triangle's side lengths and explicitly calculated their squares, showing not only procedural fluency but also a growing conceptual awareness of the Pythagorean relationship. His explanation- based on the computation of areas rather than lengths–shows that he was operating within the *process phase,* coordinating the concepts of side, square, and area in a unified structure.

In contrast, later; S11 calculated the area of a square using additive reasoning (e.g., summing sides), which S1 immediately corrected, emphasizing that area should be found by squaring the side length. This moment highlights how *peer misconceptions can serve as cognitive triggers*, allowing students like S1 to clarify and defend their own understanding–a key feature of conceptual maturation. Although minor misconceptions about unit size persisted early on, S1 was able to revise them through scaffolded support and peer dialogue. His consistent use of area-based reasoning, coupled with the ability to correct prior errors, positions him firmly within the process stage of APOS, and suggests a possible movement toward the *object phase*, as he begins to treat squared quantities as stand-alone mathematical objects that can be generalized and reused across problems.

As the activity progressed, S1 and peers began demonstrating coordination of the concepts of square, area, and multiplication. They attempted to calculate the number of tiles by squaring the side lengths. This transition was supported by the physical manipulatives and the collaborative environment intentionally built into the task structure. The real-life connection and concrete materials were designed to guide students toward interiorization–thus fostering a shift to the *process phase*. S1's consistent reasoning and his correction of peer misconceptions further highlight how the problem design, especially the sequencing of questions and visual cues, scaffolded the APOS progression. The design implication here is that carefully placed prompts and teacher questioning are essential for students to reconstruct and refine mathematical understanding. In future iterations, clearer modeling instructions may prevent early overuse of materials or visual errors.

Group work–Object phase (S1)

The following exchange between S1 and the researcher illustrates a critical moment in S1's conceptual development:

T: In part (b), what does it say? Each area will be tiled with a single-color pattern. For example, consider these units colors as the pattern–you will use only orange here, yellow here, and blue here. But what if all the tiles that arrived are the same color? You still have to use them all. You can't return any. You have 25 tiles. What can you cover?

S1: Only this one (points to the square with side 5).

T: And how many tiles are needed for that one?

S1: 25.

T: So, you'd be able to cover just one seating area.

S1: We could cover these two (points to the two smaller squares).

T: Why? Do those two add up to 25 as well?

S1: Yes.

T: Think a little more–so you say this one is 25, and the sum of those two is 25 too?

S1: There's a triangle equality or inequality... something like that going on here.

T: If you got 25 tiles, what's the most efficient way to use them?

S1: I'd tile those two.

T: That makes sense. So what kind of relationship have you found between this triangle and these squares?

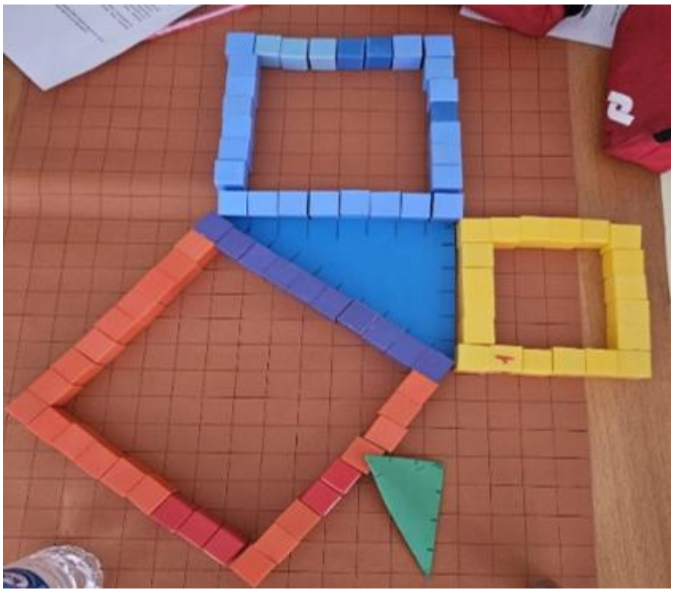


Figure 4. S1 and his group's second model showing square areas built along the sides of a 6-8-10 triangle (Source: Field study)

S1: The sum of the shorter sides equals the longer side's... thing.

T: Not the sum of the sides. What did we add?

S1: The areas.

T: Exactly–you squared the side lengths and added them, right?

S1: Yes.

T: Good. Now try it with another triangle. Let's see if it works again.

Although S1 initially struggles to verbalize the relationship precisely, his statement that "the sum of the shorter sides equals ..." and later correction toward "areas" indicates he is transitioning into the *object phase (APOS-3)*. At this stage, students begin to treat mathematical relationships–such as the Pythagorean theorem–as abstract objects that can be manipulated, tested, and generalized. S1 and his peers then selected a right triangle with side lengths 6, 8, and 10 units. They constructed square seating areas along each side and filled them with unit tiles. The completed model is shown in **Figure 4**.

Following this model, S1 called the teacher and stated:

S1: This one's 100 … this one's 64 … and this one's 36 … Teacher, they match!

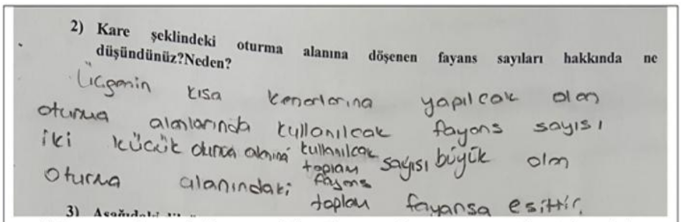
T: What do you mean?

S1: 36 + 64 = 100. So if we only received half the tiles–100 instead of 200–we could still tile two areas. These two add up to the third (points to the two squares on the legs and then to the hypotenuse square).

Here, S1 accurately calculates the areas: $6^2 = 6 \times 6 = 36$; $8^2 = 8 \times 8 = 64$; $10^2 = 10 \times 10 = 100$, 36 + 65 = 100, 100 + 100 = 200 → Total: 200 tiles. He also states that if only 100 tiles are available, it would be most efficient to tile the two smaller squares, confirming that their total equals the area of the largest square. He clearly articulates the *Pythagorean relationship* in general terms, without reliance on specific numbers. When prompted by the teacher to explain the connection between side lengths and square areas, S1 and his peers responded that in right triangles, "the sum of the squares on the legs equals the square on the side opposite the right angle." This is a textbook-level generalization and a clear indicator of *object-level understanding* (APOS-3).

At this point, S1 is no longer merely performing operations; he is treating the Pythagorean theorem as a *mathematical object*–a generalized, transferrable structure. *His ability to confirm the relationship across different triangle configurations (3-4-5 and 6-8-10), and to recognize proportional scaling between them, demonstrates abstraction and the capacity to manipulate the theorem in novel situations.* S1 further observed that the 6-8-10 triangle is a scaled version of the 3-4-5 triangle. He explicitly stated that the larger triangle was a multiple of the smaller one. This reveals his effort to *validate a mathematical property* across structurally similar cases, illustrating cognitive flexibility and abstraction- hallmarks of the object stage in APOS.

In the final phase, S1 generalized the Pythagorean theorem across two triangle configurations (3-4-5 and 6-8-10). He was able to verify that the sum of the areas of squares on the legs equaled the area on the hypotenuse and articulated this as a generalized relationship. This abstraction demonstrates object-level understanding (APOS-3), and it also serves as evidence that the instructional design achieved its aim of promoting mathematical generalization. The decision to include multiple triangle



"The number of tiles used for the seating areas on the triangle's shorter sides is equal to the number used on the larger side."

Figure 5. S1's written response during the individual interview (Source: Field study)

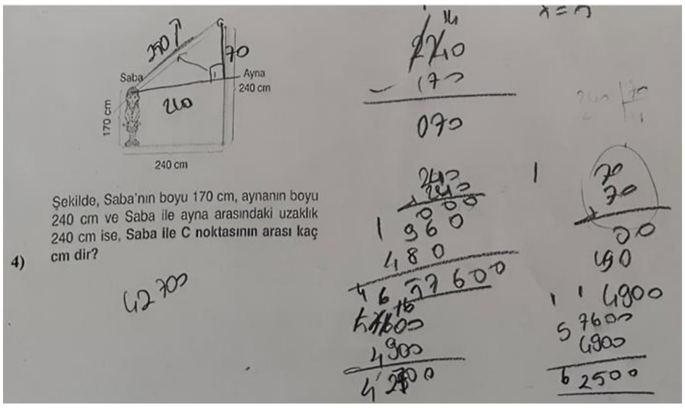


Figure 6. S1's written solution showing the application of the Pythagorean theorem (Source: Field study)

configurations and require students to evaluate tile usage under constraints was an intentional DBR design strategy to support abstraction. S1's recognition of the proportional scaling between triangles confirms that students were able to manipulate mathematical objects in novel contexts–fulfilling the higher-level cognitive goals of the task. This finding suggests that such task designs could be refined further by adding reflection prompts or representational comparisons, to deepen students' engagement with the mathematical structure.

Individual interview–Cognitive development of S1

Action phase: During the individual interview, S1 was asked to explain what he understood from the task and how he approached it. Initially, he stated that he thought the square seating areas were to be constructed inside the triangle. However, he later realized this was a misinterpretation and corrected his thinking by using units to model the expected structure. This confirms that, even in the individual setting, S1 progressed from the *action stage (APOS-1)* to the *process stage (APOS-2)* by internalizing the need to construct squares along the sides of the triangle.

Process phase: S1 reported that in order to determine how many tiles were needed, he recognized he had to square the side lengths of the triangle. This indicates a clear coordination with the mathematical concept of squaring–one of the defining features of the *process phase (APOS-2)*. When modeling, he initially repeated the earlier group mistake by placing a material at the corner where the triangle's legs meet. However, with researcher guidance, he identified the error and corrected it by calculating the areas of the squares built on the triangle's sides. In doing so, he actively engaged with related concepts such as *area*, *squaring*, and *square root*, solidifying his process-stage reasoning.

Object phase: When asked how to calculate the total number of tiles needed for the seating areas, S1 clearly stated that the total number of tiles required for the two smaller square areas (on the triangle's legs) would equal the number required for the larger square (on the hypotenuse). This response illustrates that S1 had *encapsulated* the mathematical relationship and could now treat it as a conceptual object–demonstrating development into the *object phase (APOS-3)*.

S1's written response from the individual interview supports this finding (see **Figure 5**).

S1 stated that he tested this relationship with different-sized right triangles and consistently obtained the same result, showing that he was engaging in *mathematical generalization*, a hallmark of object-level understanding. For instance, when solving a problem involving the Pythagorean theorem, he computed the squares of the legs, summed them, and then took the square root to find the hypotenuse. This shows that he had successfully *constructed and internalized* the Pythagorean relationship as an abstract mathematical object. Additionally, S1 recognized that the 3-4-5 triangle and the 6-8-10 triangle were proportional. He described them as "basically the same shape but scaled up" and noted that they seemed to "have the same properties." This observation not only reinforces his grasp of the Pythagorean theorem but also suggests the *emergence of the similarity concept* through his independent reasoning–further evidence of his progression into the object stage.

He correctly formed a right triangle, squared each leg, summed the squares, and found the hypotenuse by taking the square root of the result (**Figure 6**).

In other problems where one leg length was unknown, S1 correctly applied the inverse of the theorem by subtracting the square of one known leg from the square of the hypotenuse and then taking the square root. In this way, he consistently demonstrated that he could flexibly apply the Pythagorean theorem in both direct and inverse formats–further confirming that he had internalized it as a manipulable mathematical object (APOS-3). S1's ability to abstract, generalize, and adapt the Pythagorean theorem across multiple contexts, including novel problem-solving tasks, illustrates that he has reached the *object phase*. His recognition of proportional relationships and the conceptual transfer from one triangle to another also suggests a readiness to transition toward schema-level understanding.

Findings for Student 2 (S2)

To provide a comprehensive view of S2's conceptual development, the analysis is divided into group work and individual interview stages, aligned with the APOS framework.

Group work–Action phase (S2)

At the beginning of the activity, S2 and group members attempted to understand the task by reading it multiple times and discussing its meaning. Initially, they misinterpreted the instruction, assuming that the square seating areas were to be placed *inside* the triangular building rather than outside. This misunderstanding is reflected in S2's question:

> **S2:** Teacher, do you mean here? Or here? (points inside and then outside the triangle).
>
> **T:** Outside. Think of it like a garden area–seating spaces will surround the triangle.

S2's physical pointing and need for clarification indicated that he had not yet formed a mental representation of the geometric structure. This reliance on external cues and teacher explanation is a clear marker of the *action phase* in APOS theory, where students operate on instructions or concrete materials rather than on internalized concepts. Following this exchange, S2 and peers began manipulating the materials and asked further questions:

> **S2:** Isn't there any measurement?
>
> **T:** There's no fixed measurement. Try it out using this triangle model.
>
> **S12:** Teacher, is one tile equal to one square?
>
> **T:** Yes, think of each material as one unit of tile.

These interactions confirm that S2 was still seeking confirmation for even basic structural assumptions. His question "Why a right triangle specifically?" also reflected uncertainty about the mathematical rationale behind the problem design. After continued teacher prompting, the group selected a triangle with side lengths 3, 4, and 5 units and began attaching square areas to each side. This marked the beginning of S2's *transition toward the process phase*, as he started to coordinate the triangle's side lengths with the square constructions. S2's shift from passive questioning to initiating a geometric model signaled emerging conceptual engagement. While still reliant on concrete materials and teacher feedback, he had begun to move beyond procedural imitation and toward meaningful coordination of spatial and numerical ideas–a hallmark of the *APOS-1 to APOS-2 transition*.

S2, like S1, initially interpreted the seating area problem through a real-world lens but was hesitant to engage fully with the manipulatives. Her participation was more observational during the early phase of the group work. This pattern of behavior highlights a common scenario in DBR settings: the need to design activities that not only provoke cognitive engagement but also ensure inclusive participation. The collaborative structure of the task eventually drew S2 into the discussion, especially when prompted by peers to measure and compare side lengths. This gradual shift from passive to active involvement suggests that the group-based modeling design helped bridge initial hesitation, a common action-phase challenge.

Group work–Process phase (S2)

S2 and their peers initially attempted to construct the prism-shaped building using units. However, S2 suggested that building just the base (i.e., one right triangle) would suffice. Once the model was completed, the researcher joined the group, and the following dialogue occurred:

> **S2:** Teacher, now I understand. We'll build square seating areas here, tile them all evenly, but with different patterns.
>
> **T:** Right. Each square can be all one color. For example, this one all yellow, that one all blue, this one all orange. But they could also be the same. What matters is consistency within each square.
>
> **S2 (to group):** You do yours dark blue, and you do yours yellow (they fill in the squares together).
>
> **T:** Now, how many tiles do you need?
>
> **S2:** 4 times 4 is 16, 3 times 3 is 9, 5 times 5 is 25. So we need 50 tiles.
>
> **T:** And how did you calculate that?

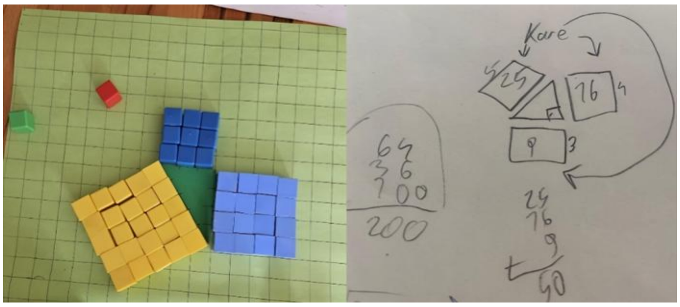


Figure 7. Model of the triangle and square seating areas constructed by S2's group (Source: Field study)

Figure 8. Model of the 6-8-10 triangle created by S2's group (Source: Field study)

> **S2:** I added them all up.
>
> **T:** Did you count the tiles or calculate something?
>
> **S2:** We counted, and also did the squares. Like, here's 4, so it must be 4 × 4.

S2's shift from counting units to squaring side lengths shows a transition to the *process phase (APOS-2)*, where the learner begins to coordinate geometric measurement (area) with numerical operations (squaring). His explanation indicates internalization of both the structure of a square and the operation required to determine its area (**Figure 7**).

In the second part of the problem, the researcher proposed a constraint: only 25 tiles (rather than 50) were available. When asked how to allocate the tiles efficiently, S2 suggested tiling the largest square only. When pushed to consider other alternatives, he struggled:

> **T:** Can you find another way? You have to use all 25 tiles, no mixing colors or leaving areas half-finished.
>
> **S2:** It won't work with 25.
>
> **T:** Try again, think carefully.
>
> **S2:** Maybe the two small squares? Then order more tiles for the big one?

While S2 demonstrated conceptual understanding in calculating total tile counts, his difficulty in adapting this understanding to constraints suggests a *partial or unstable transition toward the object phase.* He can compute areas procedurally but has not yet fully encapsulated the relationships between them as manipulable objects. Seeking a clearer perspective, S2 proposed modeling a larger triangle with sides 6, 8, and 10 (**Figure 8**). He and the group constructed the triangle and began calculating:

> **S2:** Most likely, the tile count will double. But let's check–64, 36, and 100. So, 200 tiles total.
>
> **T:** Good. And what if you received only 100 tiles? How could you tile the space?
>
> **S2:** We could tile this big one. Or maybe both small ones and then order more?

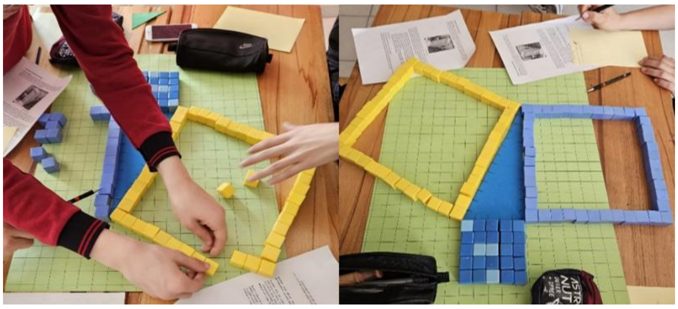


Figure 9. S2's model of the 5-12-13 triangle and square seating areas (Source: Field study)

S2 noted the similarity between the 3-4-5 and 6-8-10 triangles, recognizing proportionality between them. This shows an emerging awareness of *mathematical generalization* -a prerequisite for transition into the object stage. The teacher then introduced a new triangle with sides 5, 12, and 13, asking if the relationship would still hold (**Figure 9**). S2 and peers modeled it, built the square frames, and filled in the interior:

S2: Okay, we made the outlines. Now let's fill them.

S13: How many materials do I need?

S2: Your square's side is 13. So, 13 × 13 = 169. Take 169 yellow units and fill it in (after completing all squares).

S2: Teacher, we got it–338 tiles total.

T: How did you calculate it?

S2: One square is 5 × 5 = 25. Another is 12 × 12 = 144. Then 13 × 13 = 169. We added them all up.

This dialogue shows that S2 had fully internalized the procedure of calculating square areas by squaring the side lengths and summing the results. His explanation is independent, structured, and mathematically coherent–indicating a stable presence in the *process phase (APOS-2)*. At this point, S2 was no longer relying on the teacher's scaffolding but was instead applying the relevant mathematical operations autonomously. The student coordinated the concepts of *exponentiation, square area, and numerical addition* within a single problem-solving schema.

S2's ability to generalize area computation across three different right triangles; 3-4-5, 6-8-10, and 5-12-13 demonstrates that he is solidifying his understanding within the *process phase (APOS-2)*.

While not yet manipulating the relationships as abstract mathematical objects, he shows clear and accurate coordination of squaring and area concepts across contexts. The transition from *action to process* in APOS is marked by the student's ability to mentally perform operations without the need for physical manipulation or step-by-step instructions. S2's fluent calculation of each square's area and integration of these results into a total count exemplifies this shift. The coordination of conceptual tools (e.g., "square of 13") with spatial reasoning (e.g., "filling in a square with units") demonstrates the development of *process-oriented thinking*. Moreover, this example reflects the *emergence of structural understanding*–S2 not only executes operations correctly but also *understands their meaning* in the context of geometric modeling. His thinking begins to extend beyond one task, laying the foundation for *future generalization*–a key pathway toward the *object phase*.

S2 demonstrated evidence of process-level reasoning during the interview, where she verbalized an understanding of the relationship between the side lengths and the square areas. The structure of the worksheet and visual representation supported her coordination of multiplication as an area model. However, she still required external prompting to maintain the relationship across different triangle configurations. This finding indicates that while the design successfully supported process-level engagement, more explicit scaffolds–such as comparing multiple solved examples–might be needed to deepen independent abstraction.

Group work–Object stage (S2)

The researcher asked S2 and their group:

You ordered 338 tiles but received only 169 of the same design. With this amount, which area(s) could you tile?

The following exchange occurred:

S2: 5-12-13, teacher.

T: So, one side is 5 units, another 12, and the hypotenuse 13. How many tiles are needed?

S2: 5^2 is 25, 12^2 is 144, 13^2 is 169. The total is 338.

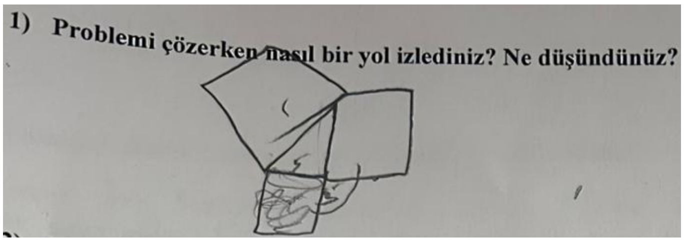


Figure 10. S2's initial drawing during individual problem-solving (Source: Field study)

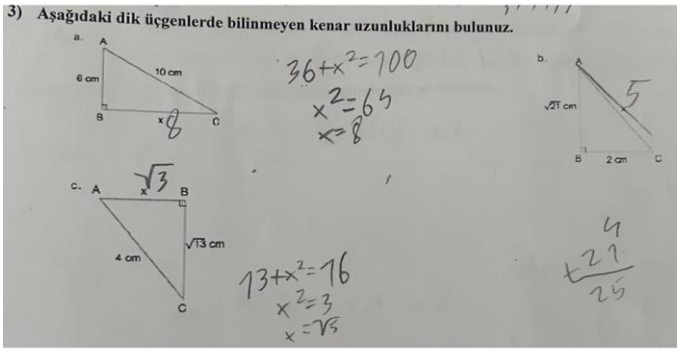


Figure 11. S2's solution to problems involving unknown triangle side lengths (Source: Field study)

T: And if only half that number of tiles arrived?

S2: 169 ... Then I could either tile this large square or the sum of the other two.

T: That's right.

S2's responses illustrate object-level reasoning within APOS theory. The student calculated the square areas corresponding to the triangle's sides, confirmed the Pythagorean relationship, and used it to reason spatially. Recognizing that 13^2 equals $5^2 + 12^2$, S2 applied this understanding to determine alternative tiling scenarios.

This shows that the student perceives the Pythagorean theorem not merely as a calculation but as a flexible concept applicable to new problems. By evaluating areas based on different tile quantities, S2 demonstrated the ability to treat the theorem as a coherent object that can be generalized. According to APOS, this indicates a shift from procedural to conceptual understanding, integrating both numerical and spatial reasoning.

S2 did not fully internalize the object stage during the interview. Her reasoning remained context-bound, and although she was able to perform correct calculations, she hesitated to articulate the general rule ($a^2 + b^2 = c^2$) independently. This outcome points to a key consideration in DBR: evaluating which elements of the design need refinement. In this case, it appears that the task could benefit from additional reflective prompts or explicit generalization phases at the conclusion of the activity. Nevertheless, S2's partial progression suggests that the learning environment laid a foundational schema that could be built upon in future iterations–an important goal of DBR cycles.

Findings from S2's individual interview

The individual interview was conducted after briefly reminding the student of the original problem using the task sheet and their group solution. Following this, conceptual and procedural questions related to the Pythagorean theorem were asked.

Action phase (S2): During the interview, S2 stated that he had relied on units, a triangle model, and grid paper while solving the problem. He noted that he first visualized the structure and translated the problem into a diagrammatic representation. His model is presented in **Figure 10**.

When asked about the number of tiles required for each square seating area, S2 said:

> I first thought I had to find the number of tiles needed for each area and then sum them.

This confirms the approach he used during the group work and validates the progression observed in his earlier responses.

Process phase (S2): S2 demonstrated a clear transition into the *process phase* by internalizing the idea that each side of a right triangle can serve as the base for a square, and that the number of tiles required corresponds to the **area** of each square. He explained:

> If one side is 5 units, then 5 times 5 makes 25 tiles. If one side is 3 units, the area is 3 times 3, so 9 tiles. And if it's 4 units, it's 4 times 4, so 16 tiles.

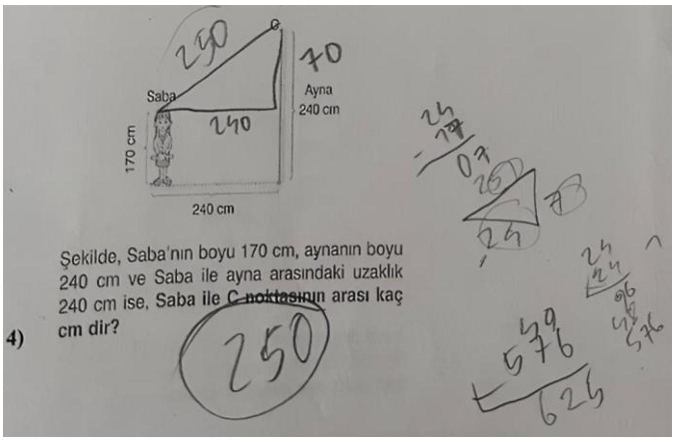


Figure 12. S2's application of the Pythagorean theorem in an independent problem (Source: Field study)

Figure 13. Initial model constructed by S3's group (Source: Field study)

This response shows coordination between the concepts of **squaring a number** and **area calculation**, consistent with the *process phase (APOS-2)*. When presented with right triangles having unknown side lengths, S2 set up and solved equations to find the missing sides. His written solution is shown in **Figure 11**.

In one case, he used the Pythagorean theorem to form an equation where the square of the known leg plus the square of the unknown leg equaled the square of the hypotenuse. Solving this equation revealed the missing length. In another case, where the hypotenuse was unknown, S2 summed the squares of the legs and found the square root of the result. In both situations, he demonstrated coordination between multiple mathematical concepts including *squaring numbers*, *solving linear equations*, and *taking square roots*. These operations, applied independently and with understanding, are strong indicators of process-level reasoning.

Object phase (S2): When asked to generalize the relationship, S2 stated:

> In right triangles, the sum of the squares of the two shorter sides is equal to the square of the longer side.

This articulation shows that S2 had encapsulated the relationship and now treated it as a mathematical object–a hallmark of the *object phase (APOS-3)*. To further assess this, S2 was given an application-based problem. He correctly constructed a right triangle, labeled the side lengths, squared the legs, summed the results, and then identified the square root of the total as shown in **Figure 12**.

S2 correctly answered all problems requiring application of the Pythagorean theorem. His consistent reasoning and ability to express the relationship as *the sum of the squares of the two shorter sides equals the square of the longer side* confirm that he had constructed and internalized the Pythagorean relationship as a **conceptual object** (APOS-3). He was no longer merely calculating; he was reasoning with an abstract mathematical principle.

Findings for Student 3 (S3)

Group work–Action phase (S3)

At the beginning of the activity, student 3 (S3) and their group attempted to build a triangular prism using unit materials, without utilizing the pre-provided triangle templates. They spent the first 20 minutes reading and interpreting the problem and began constructing a structure similar to **Figure 13**.

Figure 14. Revised model by S3's group after correction (Source: Field study)

S8: So how many floors are we building?

S3: We don't need floors–we're not building the actual house. Let's just construct the squares here.

S13: Look, I made a square–it's 6 by 6.

S8: That's wrong. There shouldn't be materials in the corners. Let's remove them.

This conversation shows that the group interpreted the task as constructing a building rather than modeling a mathematical relationship. Their focus on vertical construction and placement errors (e.g., adding extra materials at triangle corners) are clear signs of the *action phase (APOS-1)*, where learners operate based on surface features and lack internalized mathematical structures. The teacher intervened to guide the group:

T: The triangle templates on your table are marked with units. Use them for alignment.

S3: Is this triangle equilateral?

T: Do you think so? Look–this side is 3 units, this one is 4, and this one is 5. Place it properly. You'll build square seating areas on each side.

S8: This side is 3.

T: Right. So you'll create a square of side 3 here.

After this clarification, the group revised their construction (**Figure 14**). S3 noticed inconsistencies in their earlier model–such as placing four units on a side that measured only three–and took initiative to correct the layout.

T: This side should be 3 units, but you've placed four materials. That's an error.

S3: Got it. Let's fix that. I've solved it now. (to group) Mark this as 5 and add five materials–this square is done.

By identifying and correcting misalignments, S3 began to engage in more meaningful reasoning. He shifted from trial-and-error behavior to constructing squares intentionally based on triangle side lengths. These actions signify his *emergent transition from the action phase to the process phase (APOS-1 → APOS-2)*, as he started to coordinate physical modeling with mathematical relationships. Although early misconceptions–such as adding extra materials at triangle vertices–indicated a lack of abstraction, S3's responsive adjustments and alignment with geometric principles demonstrate *growing conceptual awareness.* His increased independence and ability to relate side lengths with square construction marks the foundation for future transitions within the APOS framework.

S3 engaged with the problem by physically manipulating the triangle and tile materials but struggled to interpret the contextual meaning of the task. His early responses suggested a reliance on trial-and-error strategies, indicating that he remained at the action stage of the APOS framework. This behavior aligns with the DBR goal of uncovering potential design weaknesses. S3's confusion suggests that the problem context or material representation might need to be adjusted to provide clearer cues or introductory scaffolding. S3's group also lacked strong peer facilitation, which may have limited its opportunity to observe or internalize more structured mathematical reasoning. This highlights the importance of group composition in design cycles.

Group work–Process phase (S3)

After constructing the model using units, the researcher visited S3's group and asked them to explain their solution. The following conversation took place:

T: How many tiles do you need?

S3: Thirty-eight.

T: Why?

S3: ...

T: How many tiles are there here?

S3: Six.

T: Six? Let's count: 1-2-3.

S3: Isn't there a wall on the outside?

T: Think of the wall as a thin cross-section–like a piece of paper.

S3: Oh, I thought it was the inside of the building.

T: So, how many tiles do you need now?

S3: Fifty.

T: How did you find that?

S3: I multiplied each side by itself, then added them.

T: Great. Now remember–it says each area must be tiled in one pattern. You can use different colors as patterns: yellow, blue, and purple for each square.

S3: Can two areas both be blue?

T: Yes, but let's say 25 tiles of the same color arrived. What could you tile with just those?

S3: We could tile just this one (points to the large square) or we could tile these two (points to the smaller squares).

T: So the number of tiles in these two small squares is equal to that of the large square?

S3: Yes.

This dialogue shows that S3 has entered the *process phase* of the APOS framework. Initially, he incorrectly interpreted the structure as representing the interior of the building, leading to an underestimated tile count. However, through teacher scaffolding, he re-evaluated his model, corrected the representation, and then calculated the tile count using appropriate geometric reasoning.

S3 multiplied the side lengths of each square by themselves–indicating he understood that the area of a square is calculated using ($a^2 = b^2 + c^2$). He then added the individual areas to reach the total tile count. This coordination of *squaring and addition* shows that he no longer viewed the sides as isolated measurements, but rather as part of a *structured spatial relationship.*

Additionally, when prompted to reason about what could be tiled with only 25 tiles, S3 correctly identified two alternative solutions:

(a) tile the largest square, or

(b) tile the two smaller squares.

This recognition that the sum of the areas of the smaller squares is equal to that of the larger square demonstrates a growing understanding of the *Pythagorean relationship*, even if not yet generalized abstractly. S3's explanation–"I multiplied each side by itself and added them"–is a clear verbal marker of *process-phase thinking*, wherein operations are no longer manually enacted but mentally encapsulated as coherent procedures.

During the individual interview, S3 began to exhibit signs of transition toward the *process phase*. He was able to explain that the number of tiles corresponds to the area of each square, and he began coordinating side length and area through multiplication. However, he needed ongoing prompts to maintain focus and consistently apply this reasoning. The one-on-one setting and think-aloud strategy facilitated this partial progression, underscoring the importance of supportive structures in design implementation. In future design iterations, including intermediate guiding questions or worked examples could help students like S3 progress more confidently beyond the trial-and-error phase.

Group work–Object phase (S3)

The following conversation occurred between the researcher and S3:

> **T:** Do you think this relationship holds true for triangles with different side lengths?
>
> **S3:** Yes, it does.
>
> **T:** How do you know?
>
> **S3:** One side of the building equals the sum of the other two.
>
> **T:** This side is 3 units, and that one is 4. So 3 plus 4 equals 7. Do you think this side is 7?
>
> **S3:** No–the sum of their squares. The square of this and this equals the square of that.
>
> **T:** Excellent. Do you think this applies to triangles with different side lengths? Try it and see.

Following this exchange, S3 and his group constructed a model of a right triangle with side lengths 6, 8, and 10 units. They confirmed that the Pythagorean relationship also held for this triangle. They then repeated the process with a 5-12-13 triangle and arrived at the same conclusion. At this point, S3 moved beyond applying a formula to a specific example. He began testing the relationship across multiple right triangles, forming and confirming a general mathematical rule. His verbal articulation–"*the sum of the squares of these two equals the square of that*"–demonstrates that he had internalized the Pythagorean relationship as an abstract mathematical object, capable of being manipulated, generalized, and transferred across contexts.

Rather than engaging with the concept purely as a set of procedures or a result of calculations, S3 was now reasoning about it structurally. He saw it as a consistent mathematical property, independent of specific numbers. This marks the full transition to the *object phase (APOS-3)*. This stage aligns with the APOS theory's description of the *object phase*, wherein the student encapsulates previously coordinated processes into a coherent whole and begins to treat mathematical ideas as manipulable objects in themselves (Arnon et al., 2014). S3's ability to generalize and apply the Pythagorean theorem across varied cases shows clear evidence of conceptual encapsulation and abstraction.

S3 did not reach the *object phase* during the scope of this study. While he was able to verify some area relationships numerically, he could not articulate or generalize the Pythagorean relationship across different triangle configurations. This outcome serves as a reminder in DBR that not all students will reach abstraction within a single design cycle, especially without peer modeling or enhanced teacher scaffolding. The data suggest that group roles and explicit reflection prompts should be embedded more intentionally in future instructional designs to support abstraction for students at this level.

Individual interview findings for student 3 (S3)

The individual interview was conducted after briefly reminding the student of the original problem using the task sheet and their group solution. Following this, conceptual and procedural questions related to the Pythagorean theorem were asked. In the individual interview with S3, it was confirmed that the student had successfully reached the *action phase (APOS-1)* and *process phase (APOS-2)*. However, it was also observed that S3 had not yet attained the *object phase (APOS-3)*. Although S3 was able to verbally express the Pythagorean relationship accurately during group work, he struggled to apply it independently when solving problems that required using the relationship.

According to APOS theory, reaching the *object phase* entails more than simply recalling or repeating a concept; the student must be able to perform actions on the concept**,** apply it flexibly in novel contexts, and treat it as a mathematical object (Dubinsky & McDonald, 2001). While S3 articulated that "the sum of the squares of the legs equals the square of the hypotenuse," he was unable to use this generalization effectively when solving related problems individually. This indicates that S3 had encapsulated the relationship verbally but had not internalized it as a manipulable object. In other words, the student could state the theorem but could not act on it mathematically.

DISCUSSION

This study set out to explore how students conceptualize the Pythagorean theorem through a context-driven instructional design rooted in RME and analyzed using the APOS theoretical framework. Drawing on DBR principles, the learning environment was iteratively constructed to provoke, capture, and support conceptual development through real-world mathematical contexts. The integration of RME and APOS offered a productive lens to examine students' reasoning across phases of mathematical abstraction.

The findings showed that while students initially struggled to understand the problem context–encountering the Pythagorean theorem for the first time–they gradually constructed meaningful understandings through structured RME tasks. Consistent with prior research (Bray & Tangney, 2016; Juandi et al., 2022; Özdemir & Üzel, 2011; Sitorus & Masrayati, 2016), the use of realistic contexts facilitated guided reinvention and progressive mathematization, supporting students' progression through APOS stages.

From the APOS perspective, the greatest difficulty for students was transitioning from the process to the object stage. This aligns with existing literature highlighting this transition as the most cognitively demanding (Borji et al., 2023a, 2023b; Dubinsky et al., 2005; Setyaningsih & Safi'i, 2022; Tiengyoo, Sotaro & Thaithae, 2024; Weller et al., 2004). While the *action phase* involved physical engagement and basic modeling, the *process phase* required students to coordinate multiplication and area concepts. Achieving the object stage indicated students' ability to generalize and manipulate the Pythagorean relationship abstractly (Arnon et al., 2014; Dubinsky & McDonald, 2001).

Among the three focal students, S1 and S2 reached object-level understanding. S1 was able to correct misconceptions and incorporate roots into reasoning, while S2 coordinated geometric modeling and algebraic manipulation. Their progress illustrates how foundational mathematical ideas–such as area, squaring, and root extraction–can be integrated within RME tasks to support abstraction. S3, in contrast, demonstrated appropriate language use and generalized thinking but struggled to apply knowledge to unfamiliar problems, suggesting a position near the upper boundary of the *process phase*.

This raises an important question about the sufficiency of the APOS model in capturing transitional cognitive states. The case of S3 resonates with discussions around the concept of "wholeness" and intermediate stages beyond process but short of object-level abstraction (Dubinsky et al., 2013; Oktac, 2022; Tall & Mejía-Ramos, 2021; Villabona et al., 2024; Weller et al., 2009, 2011). An expanded framework such as APTOS may offer improved explanatory power in future research.

The emergence of similarity reasoning during the triangle task–such as students recognizing the 3-4-5 and 6-8-10 triangles as "enlarged" versions–demonstrates how RME tasks can activate related geometric concepts. These incidental but meaningful discoveries support Van de Walle's (2007) and Clements and Battista's (1992) emphasis on interconnected geometric understanding within real-world contexts.

Some students also encountered difficulties with verbal reasoning and expression, particularly in the early stages. This suggests the need for instructional designs that explicitly support the development of students' mathematical language while remaining grounded in contextual understanding. As students' informal strategies evolved into generalized models, the design functioned as both a cognitive scaffold and a diagnostic tool, consistent with the principles of RME. In sum, this study illustrates the synergy between RME and APOS in capturing and supporting students' mathematical development. It emphasizes the value of realistic, well-structured problems in cultivating abstract reasoning and highlights the importance of thoughtful instructional design in bridging conceptual gaps and enabling student understanding through guided reinvention.

Implications and Recommendations

This study offers several practical and theoretical implications for mathematics educators, curriculum designers, and educational researchers working within the framework of RME and APOS theory.

For teachers

The findings emphasize the importance of designing instructional tasks that allow students to explore mathematical ideas through real-world contexts. Teachers are encouraged to incorporate structured yet open-ended tasks that support guided reinvention and scaffold transitions between APOS stages. Classroom strategies such as think-aloud interviews, small-group problem-solving, and the use of physical manipulatives can effectively support learners with diverse conceptual readiness.

For curriculum designers

The study highlights the need for curriculum materials that blend conceptual depth with accessibility. RME-aligned resources should provide opportunities for informal modeling, visual reasoning, and the gradual abstraction of mathematical ideas. Including prompts that encourage reflection and connection across representations can enhance student engagement and retention.

For researchers

The findings reinforce the value of combining RME with APOS to gain insight into students' learning processes. Future studies may explore hybrid frameworks such as APTOS to more precisely describe cognitive transitions. Additionally, longitudinal research can track how initial conceptualizations evolve into robust, abstract knowledge over time.

Instructional design recommendations

1. Design tasks that naturally elicit key mathematical relationships (e.g., side-area-square-root connections).
2. Embed variation and symmetry to stimulate recognition of related concepts (e.g., similarity).
3. Provide space for informal reasoning to surface and evolve through structured support.
4. Facilitate classroom discourse that fosters articulation and negotiation of meaning.

These recommendations aim to guide the development of learning environments that not only foster understanding of the Pythagorean theorem but also cultivate broader mathematical thinking and reasoning skills.

CONCLUSION

This study explored how eighth-grade students construct the Pythagorean theorem through a context-driven task, analyzed via the APOS framework and developed through a DBR approach. The integration of RME principles into the learning design offered meaningful entry points for students to transition through APOS stages.

Key findings demonstrated that while some students (e.g., S1) were able to reach object-level understanding of the Pythagorean theorem, others (S2 and S3) required more structured support and scaffolding. The real-world context, visual representations, and peer interactions served as important mediators in students' conceptual development.

The use of DBR enabled continuous reflection on the strengths and limitations of the learning environment. It also guided the refinement of instructional elements such as task design, scaffolding techniques, and group composition strategies. This research contributes to mathematics education by illustrating how theoretical frameworks like APOS and RME can be practically applied within an iterative design process. It emphasizes the importance of creating tasks that not only support learning goals but also reveal opportunities for further instructional improvement.

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Ethical statement: The authors stated that the study was conducted in accordance with the ethical standards of Hacettepe University. Formal approval from an institutional ethics committee was not required, as the data were collected within the scope of regular classroom practices. The data were obtained in students' natural learning environments, during their own mathematics lessons, and were collected by their regular classroom teachers as part of routine instructional activities. Participation was voluntary, and informed consent was obtained prior to data collection. All data were handled confidentially, and no personal or identifiable information was disclosed. The study complied with established ethical principles regarding participant privacy, confidentiality, and responsible data use.

AI statement: The authors stated that the manuscript was written in English by non-native speakers. For the purpose of improving the clarity and fluency of the text, we utilized artificial intelligence-based language support tools, including ChatGPT and DeepL. These tools were used solely for language editing and did not contribute to the generation of research content or analysis. All interpretations, findings, and academic arguments are entirely those of the authors.

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APPENDIX A: CONTEXTUAL PROBLEM

Realistic Life Problem

The Flatiron Building is located in Manhattan, New York, in the USA and is considered a major groundbreaking skyscraper at the time of its construction. In 1966, it was named a New York landmark. The building was added to the National Register of Historic Buildings in 1979 and was designated a National Historic Landmark in 1989.

Flatiron Building

Our mathematics teacher's grandfather worked as a contractor in New York for many years. When he returned to his country, he planned to build an apartment building in the shape of a right triangular prism, similar to the Flatiron Building, which he admired, but he died before realizing his dream. Our math teacher decided to fulfill his grandfather's dream and asked his students for help by giving them the following information.

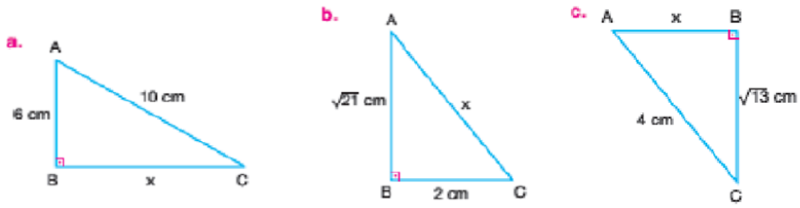
1. Three square seating areas will be constructed on three sides of the apartment building, and the remaining parts will be planted as green areas.
2. The parts planned as seating areas will be laid with square tiles of equal size. The patterns of the tiles in each seating area must have the same pattern within themselves, and there is no requirement that the tiles in different areas have different patterns.
3. Since the ordered tiles are specially designed for the order, it is impossible to return them, and they cannot be reproduced from the tiles with the same pattern again. However, tiles with different patterns can be ordered again.

According to this information,

1. How can the number of tiles to be ordered be determined?
2. If it is seen that the number of tiles received is all in the same pattern but half the number of tiles ordered, what can be done on condition that all tiles are used? How many different situations can there be?

APPENDIX B: INTERVIEW QUESTIONS

1. How did you go about solving the problem? What did you think about?
2. What did you think about the number of tiles laid in the square seating area? Why?
3. Find the unknown side lengths of the following right triangles.



4. The length of Saba is 170 cm. The length of the mirror is 240 cm. The distance between Saba and the mirror is 240 cm. What is the distance between Saba and point C?